Notes

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1 Fourier transform and L^p spaces

For a function in $f \in L^1(\mathbf{R}^n)$ define the Fourier transform

$$\hat{f}(\xi) = \int\limits_{\mathbf{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

Properties

1. $\widehat{f * g} = \hat{f}\hat{g}$
2. $\widehat{\delta_\lambda(f)}(\xi) = \hat{f}(\lambda\xi)$, where $\delta_\lambda f(x) = \lambda^{-n} f(x/\lambda)$.
3. $\widehat{\tau_h(f)}(\xi) = \hat{f}(\xi) e^{2\pi i \langle h, \xi \rangle}$, where $\tau_h(f)(x) = f(x+h)$.
4. $\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$.
5. $-\widehat{2\pi i x_j} f(\xi) = \frac{\partial}{\partial \xi_j} \hat{f}(\xi)$
6. $\widehat{fg} = \hat{f} * \hat{g}$.

Plancherel's formula

$$\int\limits_{\mathbf{R}^n} f \hat{g} dx = \int\limits_{\mathbf{R}^n} \hat{f} g dx$$

$$\int\limits_{\mathbf{R}^n} f \bar{g} dx = \int\limits_{\mathbf{R}^n} \hat{f} \bar{\hat{g}} dx, \qquad \|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

Inversion formula

$$f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$$

Schwartz class and distributions

ϕ is in the Schwartz class $\mathcal{S}$, if *all* of the seminorms

$$p_{\alpha,\beta}(\phi) = \sup_x |x^\alpha D^\beta \phi(x)| = \sup_x |x_1^{\alpha_1} \dots x_n^{\alpha_n} \partial_1^{\beta_1} \dots \partial_n^{\beta_n} \phi(x)|$$

are finite, the distributions are bounded (in the topology of $\mathcal{S}$) linear functionals on $\mathcal{S}$.

Operations on distributions

1. $\langle \partial^\alpha h, \phi \rangle := (-1)^{|\alpha|} \langle h, \partial^\alpha \phi \rangle$

2. $\langle \hat{h}, \phi \rangle := \langle h, \hat{\phi} \rangle$

3. $\langle hg, \phi \rangle := \langle h, g\phi \rangle$, as long as $g \in C^\infty$ with polynomial bounds.

4. $\langle h * g, \phi \rangle := \langle h, \tilde{g} * \phi \rangle$, where $\tilde{g}(x) := g(-x)$.

Homogeneous Distributions

$$\widehat{(|x|^{-a})}(\xi) = \frac{\pi^{a-n/2} \Gamma((n-a)/2)}{\Gamma(a/2)} |\xi|^{a-n}.$$

The fundamental solution of the Laplace's operator $G(X)$ (i.e. $\Delta G = \delta$) is

$$G(X) = \begin{cases} \frac{\Gamma(n/2)}{2(2-n)\pi^{n/2}} |x|^{2-n} & n \geq 3, \\ \frac{1}{2\pi} \ln(|x|) & n = 2 \end{cases}$$

Explicit solution to some notable PDE's

The solution to the Helmholtz equation

$$\Delta u = f$$

is given by

$$u(x) = \begin{cases} -\frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \int \frac{f(y)}{|x-y|^{n-2}} dy & n \geq 3 \\ (2\pi)^{-1} \int f(y) \ln(|x-y|) dy & n = 2 \end{cases}$$

The solution to the heat equation

$$u_t = \Delta u; \quad u(0,x) = f(x), \qquad x \in R^n$$

is given by

$$u(t,x) = \frac{1}{(4t\pi)^{n/2}} \int_{R^n} e^{-|x-y|^2/4\pi t} f(y)dy.$$

The solution to the Schrödinger equation

$$iu_t = \Delta u; \quad u(0,x) = f(x), \qquad x \in R^n$$

is given by

$$u(t,x) = \frac{1}{(4it\pi)^{n/2}} \int_{R^n} e^{i|x-y|^2/4\pi t} f(y)dy.$$

Approximation of identity

Lemma 1. *Suppose* Φ *is sufficiently decaying and* $\int \Phi dx = 1$*, then*

$$\lim_{t\to 0} \|\Phi_t * f - f\|_{L^p} = 0$$

for all $1 \le p < \infty$*.*

2 Interpolation

Complex Interpolation

Theorem 1. *(Riesz-Thorin interpolation theorem) Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$*. Suppose that for some measure spaces* (Y,ν)*,* (X,μ)*, and a linear operator* T

$$\|Tf\|_{L^{q_0}(Y,\nu)} \le C_0 \|f\|_{L^{p_0}(X,\mu)}, \quad i.e.\ T : L^{p_0}(X,\mu) \to L^{q_0}(Y,\nu),$$
$$\|Tf\|_{L^{q_1}(Y,\nu)} \le C_1 \|f\|_{L^{p_1}(X,\mu)}, \quad i.e.\ T : L^{p_1}(X,\mu) \to L^{q_1}(Y,\nu).$$

Then for all $\theta \in (0,1)$ *and* $(p,q) : 1/p = (1-\theta)/p_0 + \theta/p_1;\ 1/q = (1-\theta)/q_0 + \theta/q_1$*, there exists a constant* C

$$\|Tf\|_{L^q(Y,\nu)} \le C\|f\|_{L^p(X,\mu)}, \quad i.e.\ T : L^p(X,\mu) \to L^q(Y,\nu)$$

where $C \le C_0^{(1-\theta)} C_1^{\theta}$*.*

Real Interpolation

Theorem 2. *(Marcinkiewicz interpolation theorem) Let* $1 \le p_0, p_1 \le \infty$*. Suppose that for a subinear operator* T

$$\|Tf\|_{L^{p_0,\infty}} \le C_0 \|f\|_{L^{p_0}}, \quad i.e.\ T : L^{p_0} \to L^{p_0,\infty},$$
$$\|Tf\|_{L^{p_1,\infty}} \le C_1 \|f\|_{L^{p_1}}, \quad i.e.\ T : L^{p_1} \to L^{p_1,\infty}.$$

Then for all $\theta \in (0,1)$ *and* $p : 1/p = (1-\theta)/p_0 + \theta/p_1$*, there exists a constant* $C = C(C_0, C_1, \theta)$*, but independent of* f*, so that*

$$\|Tf\|_{L^p} \le C\|f\|_{L^p}, \quad i.e.\ T : L^p \to L^p.$$

Hausdorf-Young inequality
For all $2 \le p \le \infty$, $\|\hat{f}\|_{L^p} \le C_p \|f\|_{L^{p'}}$, where $1/p + 1/p' = 1$.
Young's inequality
For all $1 \le p, q, r \le \infty$ and $1 + 1/q = 1/p + 1/r$,

$$\|f * g\|_{L^q} \le \|f\|_{L^p} \|g\|_{L^r}.$$

Moreover, if in addition $p > 1$, $1 < q, r < \infty$, then

$$\|f * g\|_{L^q} \le \|f\|_{L^{p,\infty}} \|g\|_{L^r}.$$

Weak L^p spaces
A function f is in the space $L^{p,\infty}$, if $\sup_{\lambda>0} \lambda^{1/p} |\{x : |f(x)| \ge \lambda\}| < \infty$.

3 Sobolev embedding theorems

Define the Sobolev spaces

$$\dot{W}^{p,s} = \{f : \||\nabla|^s f\|_{L^p(\mathbf{R}^n)} < \infty\}$$
$$W^{p,s} = \dot{W}^{p,s} \cap L^p$$

Lemma 2. *(Non-sharp version)*
For all $1 \le p < q \le \infty$ *with* $n(1/p - 1/q) < s$*,*

$$\|f\|_q \le C_{q,p} \|f\|_{W^{p,s}}.$$

Lemma 3. *(Sharp version)*
For all $1 < p < q < \infty$ *with* $s = n(1/p - 1/q)$*,*

$$\|f\|_q \le C_{q,p} \|f\|_{\dot{W}^{p,s}}.$$

4 Hardy-Littlewood maximal function and singular integrals

The (uncentered) Hardy-Littlewood maximal function is defined by

$$\mathcal{M}f(x) = \sup_{Q \supset x, Q-cube} \frac{1}{|Q|} \int\limits_Q |f(y)| dy.$$

We have that in pointwise sense $\mathcal{M}_{cubes} f \sim \mathcal{M}_{balls} f \sim \mathcal{M}_{dyadic\ balls} \sim \mathcal{M}_{dyadic\ cubes}$.

Theorem 3. *The Hardy-Littlewood maximal function* $\mathcal{M}f$ *is bounded on* L^p*,* $1 < p \le \infty$ *and is of weak type* $(1,1)$*, that is*

$$\|\mathcal{M}f\|_p \le C_p \|f\|_p$$
$$\sup_\lambda \lambda |\{x : \mathcal{M}f(x) \ge \lambda\}| \le C \|f\|_1.$$

Lemma 4. *Let* ϕ *be such that there exists an integrable, radial decreasing majorant* ψ*. Then*

$$\sup_\varepsilon |f * \phi_\varepsilon(x)| \le \|\psi\|_{L^1} \mathcal{M}f(x).$$

Definition 1. *We say that an operator given by* $Tf(x) = \int K(x,y) f(y) dy$ *for all* $x \notin supp(f)$ *is* **Calderón-Zygmund** *if they satisfy*

1. $T : L^2 \to L^2$

2. $\sup_{y,z} \int_{|x-y|\geq 2|y-z|} |K(x,y) - K(x,z)|dx \leq C$, *which is known as the* **Hörmander's condition**.

- *Condition* 1 *above is equivalent to* $\hat{K} \in L^\infty$, *when* $K(x,y) = K(x-y)$.

- *The Hörmander's condition is a consequence of the easier to verify* **gradient condition**, *when* $K(x,y) = K(x-y)$. *which is* $|\nabla K(x)| \leq C|x|^{-n-1}$.

Theorem 4. *The Calderón-Zygmund singular integrals are bounded operators on* L^p *and are of weak type* $(1,1)$, *that is*

$$\|Tf\|_p \leq C_p \|f\|_p$$
$$\sup_\lambda \lambda|\{x : |Tf(x)| \geq \lambda\}| \leq C\|f\|_1.$$

Examples

- The Hilbert transform is

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

  It is also given by $\widehat{Hf}(\xi) = -isgn(\xi)\hat{f}(\xi)$, whence $H^2 = -Id$.

- The Riesz transforms are defined by

$$R_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy$$

  with $\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, whence $\sum R_j^2 = -Id$.

Corollary 1. *We have for all* $1 < p < \infty$, *that*

$$\|f\|_{L^p(R^1)} \sim \|Hf\|_{L^p(R^1)}$$
$$\|f\|_{L^p(R^n)} \sim \sum_{j=1}^n \|R_j f\|_{L^p(R^n)}$$

Also, for every $1 < p < \infty$ and all integers s,

$$\||\nabla|^s f\|_{L^p(R^n)} \sim \sum_{\alpha:|\alpha|=s}^{n} \|\partial^\alpha f\|_{L^p(R^n)}$$

In particular, any of these two expressions can be taken for a norm in $\dot{W}^{p,s}$.

Corollary 2. *For the Helmholtz equation, $\Delta u = f$, one has*

$$\sup_{j,k=1,\dots,n} \|\partial_j \partial_k u\|_{L^p(R^n)} \le C_n \|f\|_{L^p(R^n)}.$$

Vector valued Calderón-Zygmund operators

Theorem 5. *Suppose $Tf(x) = \int K(x-y)f(y)dy$, where for every x, $K(x)$ is a bounded operator between two Banach spaces A and B, that is $K(x) \in L(A,B)$. If T satisfies*

1. $T : L^2(A) \to L^2(B)$,
2. $\|\nabla K(x)\|_{L(A,B)} \le C|x|^{-n-1}$.

Then $T : L^p(A) \to L^p(B)$ and is of weak type $(1,1)$, that is

$$\left(\int \|Tf(x)\|_B^p dx\right)^{1/p} \le C_p \left(\int \|f(x)\|_A^p dx\right)^{1/p}$$

$$\sup_\lambda \lambda |\{x : \|Tf(x)\|_B \ge \lambda\}| \le C \int \|f(x)\|_A dx.$$

Corollary 3. *Let $T : L^2 \to L^2$ and satisfies the Hörmaner's condition. Then for every $1 \le r \le \infty$,*

$$\| \left(\sum_j |Tf_j|^r\right)^{1/r} \|_{L^p(R^n)} \le C_n \| \left(\sum_j |f_j|^r\right)^{1/r} \|_{L^p(R^n)}$$

5 Littlewood-Paley theory

Define a function $\Phi \in C_0^\infty(\mathbf{R}^n)$, so that Φ is even and supp $\Phi \subset \{\xi : |\xi| \leq 2\}$ and $\Phi(\xi) = 1, |\xi| \leq 1$. Take $\psi(\xi) := \Phi(\xi) - \Phi(2\xi)$. Note $\sum_{k\in\mathcal{Z}} \psi(2^{-k}\xi) = 1$. Define

$$\widehat{P_k f}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi)$$
$$P_k f(x) = 2^{kn} \int_{\mathbf{R}^n} \check{\psi}(2^k(x-y))f(y)dy$$
$$P_{\leq k} = \sum_{m\leq k} P_m \qquad etc.$$

Proposition 1. *For all* $1 \leq p \leq \infty$:

$$c_1 \sup_k \|P_k f\|_p \leq \|f\|_p \leq \sum_k \|P_k f\|_p.$$

For all $1 \leq p \leq \infty$:

$$\|\nabla P_k f\|_p \sim 2^k \|P_k f\|_p.$$

5.1 Bernstein inequality & Sobolev embedding

Proposition 2. *(Bernstein) Let* $A \subset R^n$ *be any measurable set,* $\widehat{P_A f}(\xi) := \chi_A(\xi)\hat{f}(\xi)$. *Let* $1 \leq q \leq 2 \leq p \leq \infty$. *Then*

$$\|P_A f\|_{L^p} \leq |A|^{1/q-1/p}\|f\|_{L^q}.$$

For Littlewood-Paley pieces $P_k f$ *this can be generalized to* $1 \leq q \leq p \leq \infty$

$$\|P_k f\|_{L^p(R^n)} \leq C_n 2^{kn(1/q-1/p)}\|P_k f\|_{L^q(R^n)}.$$

5.2 Littlewood-Paley theory - Square functions

Theorem 6. *For all* $1 < p < \infty$*, one has*

$$\left\| \left(\sum_{k\in\mathcal{Z}} |P_k f|^2 \right)^{1/2} \right\|_p \sim \|f\|_p.$$

The dual inequality for an arbitrary sequence $\{f_k\}_k$ (maybe not Littlewood-Paley pieces!)

$$\left\|\sum_k P_k f_k\right\|_p \leq C_p \left\|\left(\sum |f_k|^2\right)^{1/2}\right\|_p.$$

5.3 Hörmander-Mikhlin multiplier criteria

Theorem 7. *Let T_m be a multiplier type operator acting by $\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi)$. Suppose that $|\nabla^j m(\xi)| \leq C_j |\xi|^{-j}, \quad j = 0, \ldots,$. Then $T : L^p \to L^p$ for all $1 < p < \infty$.*

6 Sobolev spaces and product estimates

6.1 Sobolev Spaces

Theorem 8.

$$\||\nabla|^s f\|_{L^p} \sim \left\|(\sum_k 2^{2sk}|P_k f|^2)^{1/2}\right\|_{L^p}.$$

6.2 Littlewood-Paley projections on products

Lemma 5. *For every two functions f, g*

$$supp\widehat{fg} \subseteq supp\hat{f} + supp\hat{g}.$$

As a consequence,

$$\begin{aligned} P_k(fg) &= P_k[P_{<k+5} f P_{k-3<\cdot<k+3} g] + P_k[P_{k-1<\cdot<k+1} f P_{\leq k-3} g] + \\ &\quad + P_k[\sum_{l \geq k+3} P_{l-2\leq\cdot\leq l+2} f P_l g]. \end{aligned}$$

6.3 Product estimates

Theorem 9. *Let $1 < p, q_1, q_2 < \infty$, $1 < r_1, r_2 \leq \infty$, $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$, $s > 0$. Then*

$$\||\nabla|^s (fg)\|_{L^p} \lesssim \||\nabla|^s f\|_{L^{q_1}} \|g\|_{L^{r_1}} + \||\nabla|^s g\|_{L^{q_2}} \|f\|_{L^{r_2}}.$$

If s is an integer, then one can replace $|\nabla|^s$ with ∇^s above.

Corollary 4. *If $s > n/p$, the space $W^{s,p}(\mathbf{R}^n)$ is a Banach algebra, that is whenever $f, g \in W^{s,p}(\mathbf{R}^n)$, then $fg \in W^{s,p}(\mathbf{R}^n)$ and*

$$\|fg\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}} \|g\|_{W^{s,p}}$$

7 Semigroups and Dynamics

For the linear homogeneous equation

$$\left| \begin{array}{l} u' + Au = 0 \\ u(x,0) = u_0(x) \end{array} \right.$$

one has a solution $x(t) = S(t)u_0 = e^{tA}u_0$ in appropriate sense. Under suitable assumptions on $A : D(A) \subset X \to X$ (which is in general unbounded) one has *the semigroup properties*

1. $S(t) \in B(X)$ and $S(t+s) = S(t)S(s)$
2. $S(0) = Id$
3. $S(t)x \to S(t_0)x$ as $t \to t_0$ for every fixed x.

Given the semigroup $S(t)$, one may recover the generator by

$$\begin{aligned} D(A) &:= \{u : \lim_{t\to 0} \frac{S(t)u - u}{t} \quad \text{exists}\} \\ Au &:= \lim_{t\to 0} \frac{S(t)u - u}{t} \end{aligned}$$

For the nonhomogeneous equation

$$\left| \begin{array}{l} u' + Au = F(x,t) \\ u(x,0) = u_0(x) \end{array} \right.$$

one solves by the **Duhamel's formula**

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,\cdot)ds.$$

7.1 Examples

1. Heat semigroup.
   Generated by $A = \Delta$, $D(A) = H^2(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$.
   $$S(t)f = e^{t\Delta} f \qquad \widehat{S(t)f}(\xi) = e^{-4\pi^2 t|\xi|^2} \hat{f}(\xi).$$
   Using the kernel representation
   $$S(t)f(x) = \frac{1}{(4t\pi)^{n/2}} \int_{R^n} e^{-|x-y|^2/4\pi t} f(y)dy.$$
   $S(t)$ is a semigroup of contractions on all $L^p(\mathbf{R}^n), 1 \leq p \leq \infty$.

2. Schrödinger semigroup.
   Generated by $A = i\Delta$, $D(A) = H^2(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$.
   $$S(t)f = e^{it\Delta} f \qquad \widehat{S(t)f}(\xi) = e^{-4\pi^2 it|\xi|^2} \hat{f}(\xi).$$
   Semigroup of unitary operators on $L^2(\mathbf{R}^n)$, **but not a semigroup on any** $L^p(\mathbf{R}^n), p \neq 2$.

3. Wave equation.
   For $u_{tt} - \Delta u = 0$, take $v := u_t$ to write in the form
   $$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$
   For the semigroup generated by $A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$,
   $$e^{tA} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{-\Delta}) & \sin(t\sqrt{-\Delta})/\sqrt{-\Delta} \\ -\sqrt{-\Delta}\sin(t\sqrt{-\Delta}) & \cos(t\sqrt{-\Delta}) \end{pmatrix}$$
   $$u(t) = S(t) \begin{pmatrix} f \\ g \end{pmatrix} = \cos(t\sqrt{-\Delta}) f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g.$$
   The semigroup $S(t) : L^2(\mathbf{R}^n) \times \dot{H}^{-1}(\mathbf{R}^n) \to L^2(\mathbf{R}^n) \times \dot{H}^{-1}(\mathbf{R}^n)$. For the inhomogeneous equation $u_{tt} - \Delta u = F(x,t)$, one has the Duhamel's formula
   $$u(t,x) = \cos(t\sqrt{-\Delta}) f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s,\cdot) ds.$$

7.2 A Local solvability theorem

Theorem 10. *Assume that there exist* $T > 0$*, Banach spaces* X_T, N_T *and* $s > 0$*, so that*

- $X \hookrightarrow C([0,T], H^s)$*, i.e.* $\|\cdot\|_{C([0,T],H^s)} \leq C\|\cdot\|_{X_T}$*,*
- $\|S(t)u_0\|_{X_T} \leq C\|u_0\|_{H^s}$ *,*
- $\left\| \int_0^t S(t-s)F(s,\cdot)ds \right\|_{X_T} \leq Co(t)\|F\|_{N_T}$*, where* $o(t) \to 0$ *as* $t \to 0$*.*
- $\|G(u)\|_{N_T} \leq C\|u\|_{X_T}^{\alpha}$ *for some* $\alpha > 1$*.*
- $\|G(u) - G(v)\|_{N_T} \leq C(\|u\|_{X_T}^{\alpha-1} + \|v\|_{X_T}^{\alpha-1})\|u-v\|_{X_T}$ *for some* $\alpha > 1$*.*

Then, there exists a time $t_0 = t_0(\|u_0\|_{H^s}) < T$*, so that in the interval* $0 \leq t \leq t_0$*, there is a unique mild solution to the Cauchy problem*

$$\left| \begin{array}{l} u' = Au + G(u) \\ u(0) = u_0. \end{array} \right.$$

That is, there exists an unique solution to the integral equation

$$u(t,x) = S(t)u_0 + \int_0^t S(t-s)F(u(s,\cdot),s,\cdot)ds.$$

Moreover, the solution has Lipschitz dependence on initial data

$$\|u-v\|_{X_T} \leq C\|u_0 - v_0\|_{H^s}.$$

7.3 Global solvability for small data

Theorem 11. *Assume that there exist Banach spaces* X, N *and* $s > 0$*, so that*

- $X \hookrightarrow C([0,\infty), H^s)$*, i.e.* $\|\cdot\|_{C([0,\infty),H^s)} \leq C\|\cdot\|_{X}$*,*
- $\|S(t)u_0\|_{X} \leq C\|u_0\|_{H^s}$ *,*

- $\left\| \int_0^t S(t-s)F(s,\cdot)ds \right\|_X \leq C\|F\|_N$.
- $\|F(u)\|_N \leq C\|u\|_X^\alpha$ *for some* $\alpha > 1$.
- $\|F(u) - F(v)\|_N \leq C(\|u\|_X^{\alpha-1} + \|v\|_X^{\alpha-1})\|u - v\|_X$ *for some* $\alpha > 1$.

Then, there exists an $\varepsilon > 0$*, so that whenever* $\|u_0\|_{H^s} \leq \varepsilon$*, there is a unique global solution to the integral equation*

$$u(t,x) = S(t)u_0 + \int_0^t S(t-s)F(u(s,\cdot),s,\cdot)ds.$$

Moreover, the solution has Lipschitz dependence on initial data, that is

$$\|u - v\|_X \leq C\|u_0 - v_0\|_{H^s}.$$

7.4 Solving nonlinear equations by fixed point methods

For the nonlinear equation

$$\left| \begin{array}{l} u' + Au = F(u,x,t) \\ u(x,0) = u_0(x) \end{array} \right.$$

produce a fixed point for the map $\Lambda : X \to X$

$$\Lambda(u) = S(t)u_0 + \int_0^t S(t-s)F(u(s,\cdot),s,\cdot)ds.$$

in appropriate space X.

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8 Strichartz estimates

Definition 2. *We say that a pair of exponents* (q,r) *is* σ *admissible, if* $q,r \geq 2$: $1/q + \sigma/r \leq \sigma/2$*. If equality holds, we say that such pair is sharp* σ *admissible.*

Theorem 12. *(Keel-Tao) Suppose that for the one parameter family of mappings $U(t)$ and for every Schwartz function f and some $\sigma > 0$*

- $\|U(t)f\|_{L^2} \leq C\|f\|_{L^2}$ *(energy estimate)*

- $$\|U(t)U(s)^*f\|_{L^\infty} \leq C|t-s|^{-\sigma}\|f\|_{L^1} \quad \text{untruncated decay estimate}$$

  or

  $$\|U(t)U(s)^*f\|_{L^\infty} \leq C(1+|t-s|)^{-\sigma}\|f\|_{L^1} \quad \text{truncated decay estimate}$$

Then under the untruncated decay estimate assumption and for every sharp σ admissible pairs $(q,r),(\tilde{q},\tilde{r})$, we have the Strichartz estimates

$$\|U(t)f\|_{L_t^q L_x^r} \leq C\|f\|_{L^2} \tag{1}$$

$$\left\|\int U(s)^*F(s,\cdot)ds\right\|_{L^2} \leq \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{2}$$

$$\left\|\int_0^t U(t)U(s)^*F(s,\cdot)ds\right\|_{L^q L^r} \leq \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{3}$$

Under the truncated decay estimate assumption the estimates above hold for every σ admissible pairs $(q,r),(\tilde{q},\tilde{r})$.

8.1 Strichartz estimates for the Schrödinger equation

Theorem 13. *For the Schrödinger equation $iu_t + \Delta u = F, \quad u(x,0) = u_0$, one has the propagator $U(t) = e^{it\Delta}$ for which one has the energy and decay estimates*

$$\left\|e^{it\Delta}f\right\|_{L^2} = \|f\|_{L^2} \tag{4}$$

$$\left\|e^{it\Delta}f\right\|_{L^\infty(\mathbf{R}^n)} \leq Ct^{-n/2}\|f\|_{L^1(\mathbf{R}^n)} \tag{5}$$

As a consequence of the abstract theorem for every $(q,r),(\tilde{q},\tilde{r}) : q,r,\tilde{q},\tilde{r} \geq 2, 1/q + n/(2r) = n/4, 1/\tilde{q} + n/(2\tilde{r}) = n/4,$

$$\left\|e^{it\Delta} f\right\|_{L_t^q L_x^r} \leq C\|f\|_{L^2} \tag{6}$$

$$\left\|\int e^{-is\Delta} F(s,\cdot)ds\right\|_{L^2} \leq \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{7}$$

$$\left\|\int_0^t e^{i(t-s)\Delta} F(s,\cdot)ds\right\|_{L^q L^r} \leq \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{8}$$

8.2 Strichartz estimates for the wave equation

Theorem 14. *For the wave equation* $u_{tt} - \Delta u = F, \quad u(x,0) = f, u_t(x,0) = g$ *and for every* $\alpha \in \mathbf{R}$ *and all* $(q,r) : q,r \geq 2, 1/q + (n-1)/(2r) \leq (n-1)/4$ *one has*

$$\left\||\nabla|^{1/q+n/r+\alpha} u(x,t)\right\|_{L^q L^r} \leq C(\|f\|_{\dot{H}^{n/2+\alpha}} + \left\||\nabla|^{1/\tilde{q}'+n/\tilde{r}'+\alpha-2} F\right\|_{L^{\tilde{q}'} L^{\tilde{r}'}} \tag{9}$$

9 Applications

9.1 Semilinear Schrödinger equations

For the semilinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-2} u = 0 \tag{10}$$

take $p = 2(n+2)/n$. This is the L^2 scale invaraint semilinear Schrödinger equation.

Theorem 15. *For* (10), *one has global well-posedness for small data in* L^2 *and local well posedness in* H^ε *whenever* $\varepsilon > 0$.

More generally, let

$$\left| \begin{array}{l} iu_t + \Delta u + g(u) = 0 \\ u(x,0) = u_0(x) \end{array} \right.$$

Theorem 16. *Let $n \geq 3$, (q, r) be Schrödinger admissible in $\mathbf{R}^n$ and $r \in [2, 2n/(n-2))$. Let $M = \max(\|u\|_{L^2}, \|v\|_{L^2})$.*

$$\|g(u) - g(v)\|_{L^{r'}} \leq K(M)(\|u\|_{L^r}^\alpha + \|v\|_{L^r}^\alpha)\|u - v\|_{L^r}$$

- *Then if $\alpha + 2 < q$ and for every $u_0 \in L^2$, there exists a time $T_{\max}$ and a unique solution $u \in C([0, T_{\max}), L^2(\mathbf{R}^n)) \cap L^q([0, T_{\max}), L^r(\mathbf{R}^n)))$.*
- *If $T_{\max} < \infty$, then $\lim_{t \to T_{\max}} \|u(t, \cdot)\|_{L^2} = \infty$ (**blowup alternative**)*
- *For every u_0, v_0, the corresponding solutions satisfy*

  $$\|u - v\|_{C([0,T_{\max}),L^2(\mathbf{R}^n))\cap L^q([0,T_{\max}),L^r(\mathbf{R}^n)))} \leq C\|u_0 - v_0\|_{L^2}$$

  Lipschitz continuity with respect to initial data
- *If in addition $Im \int g(\phi)\bar{\phi}dx = 0$ for all Schwartz functions ϕ, then the solutions are global (i.e. $T_{\max} = \infty$) and there is the energy conservation law $\|u(t, \cdot)\|_{L^2} = \|u_0\|_{L^2}$.*

9.2 Wave maps

This is a system of equations in the form

$$\left| \begin{array}{l} \Box\phi + a\partial\phi = 0 \\ \Delta a \sim \partial a^2 + \partial((\partial\phi)^2) \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{array} \right.$$

Theorem 17. *In dimension $n \geq 4$, the wave map system is globally well-posed for small data $(f, g) \in \dot{H}^{n/2} \times \dot{H}^{n/2-1}$.*

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