Notes MATH 800

May 3, 2021

1 Cauchy-Riemann theory

We have covered the "Fundamental concepts", Chapter I, sections 1.1, 1.2. In particular, we are identifying $\mathbb{C} \simeq \mathbb{R}^2$. Then, we introduced holomorphic functions and derived the Cauchy-Riemann (CR) equations, section 1.4. For each holomorphic function $f = u + iv$, we have that

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases} \tag{1}$$

In the process, we discussed the relation between the derivatives $\partial_z, \partial_{\bar{z}}$ and the regular derivatives ∂_x, ∂_y. In particular, CR equations imply that both u, v are harmonic functions, which impacts the subject in a dramatic way, and also relates it to classical PDE theory.

The main result that we needed for Section 1.5, is an extension of Theorem 1.5.1.

Definition 1. *We say that a set $\Omega \subset \mathbb{C}$ is connected (or path-connected), if for any two points z_0, z_1, there exists a curve γ in Ω connecting the two points, that is, a map $\gamma : [0,1] \to \Omega$, so that $\gamma(0) = z_0, \gamma(1) = z_1$. Open and connected sets are called domains.*
We say that Ω is simply connected, if the interior of each closed curve in Ω belongs to Ω.

Then, we have the following useful result.

Theorem 1. *Let $\Omega \subset \mathbb{C}$ is a simply connected open set. Suppose that F, G are two $C^1(\Omega)$ functions on it. Then, the following are equivalent*

- $F_y = G_x$

- *There exists unique (up to a constant) function $u \in C^2(\Omega)$, so that $u_x = F, u_y = G$.*

We showed by an example that simple connectedness is a necessary condition for Theorem 1 to hold. We used Theorem 1 to derive some further useful results, like every holomorphic function in a simply connected domain has a holomorphic antiderivative (extension of Theorem 1.5.3). Also, for every harmonic function u, there is a holomorphic function F, unique up to a constant, so that $\Re F = u$ (extension of Corollary 1.5.2).

We discussed line integrals in the complex plane setting. In particular, we have introduced integral over curve as follows - for every continuous function f on an open set $U \subset \mathbb{C}$, and for a smooth curve γ in U,

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 f(\gamma(t))(\gamma_1'(t) + i\gamma_2'(t))dt.$$

where $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ is a particular parametrization. One then has the following formula.

Proposition 1. *Let U open and $f : U \to \mathbb{C}$ be a holomorphic mapping, and $\gamma : [a,b] \to U$ is a curve. Then,*

$$\int_\gamma \frac{\partial f(z)}{\partial z} dz = f(\gamma(b)) - f(\gamma(a)).$$

Note that while this lemma looks deceptively similar to the fundamental theorem of calculus, **it requires $f \in H(U)$, and not just $f \in C^1(U)$.**

An interesting corollary is the Cauchy theorem (we prove it later in detail).

Corollary 1. *(Cauchy theorem) Let Ω be a simply connected domain, $\gamma : [0,1] \to \Omega$ be a closed curve and $f \in H(\Omega)$. Then*

$$\int_\gamma f(z)dz = 0$$

Proof. We have shown that there is $F \in H(\Omega) : f = \frac{\partial F}{\partial z}$, see the extension of Theorem 1.5.3 to simply connected domains. Then, by Proposition 1,

$$\int_\gamma f(z)dz = \int_\gamma \frac{\partial F}{\partial z} dz = F(\gamma(1)) - F(\gamma(0)) = 0.$$

□

Proposition 2. *Let* $\phi : [a,b] \to \mathbf{C}$ *be a continuous function. Then,*

$$|\int_a^b \phi(t)dt| \le \int_a^b |\phi(t)|dt.$$

Proposition 3. *Let* $\gamma : [a,b] \to \mathbf{C}$ *be a* C^1 *curve. Then the curve length of* γ *is*

$$l(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}dt.$$

We have the following proposition.

Proposition 4. *Let* $\gamma : [a,b] \to U \subset \mathbf{C}$ *be a* C^1 *curve,* $f : U \to \mathbf{C}$ *be continuous function. Then,*

$$|\int_\gamma f(z)dz| \le \max_{z\in\gamma} |f(z)| l(\gamma).$$

Another remark is that the particular parametrization of γ does not affect the value of the integral, as long as the curve is traced in the same direction. More precisely,

Proposition 5. *Let* $\gamma : [a,b] \to U \subset \mathbf{C}$ *be a* C^1 *curve,* $f : U \to \mathbf{C}$ *be a continuous function. Assume* $\phi[0,1] \to [a,b]$ *be a monotone and one-to-one function. Then,* $\tilde{\gamma} := \gamma \circ \phi : [0,1] \to U$ *is also a curve and* $\int_{\tilde{\gamma}} f(z)dz = \int_\gamma f(z)dz$ *if* ϕ *is increasing, and* $\int_{\tilde{\gamma}} f(z)dz = -\int_\gamma f(z)dz$ *if* ϕ *is decreasing.*

Definition 2. *Let* U *be an open set in* $\mathbb{C}$*. Let* $f : U \to \mathbb{C}$ *be a function. Then, we say that* f *is complex differentiable at* $z_0 \in U$*, if the limit exists*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

We denote the limit by $f'(z)$

It turns out that this notion is equivalent with holomorphicity. In fact, we have the following equivalence theorem.

Theorem 2. *Let* $U \subset \mathbb{C}$ *is an open set and* $f \in C^1(U)$*. Then,* f *is holomorphic on* U *if and only if* $f'(z)$ *exists for all* $z \in U$*. In addition,*

$$f' = \frac{\partial f}{\partial z}$$

Finally, we considered a "one point" extension of Theorem 1. More specifically,

Theorem 3. *Let $\Omega \subset \mathbb{C}$ is a simply connected open set, $P \in \Omega$. Suppose that f, g are two continuous functions on Ω, with continuous derivatives in $\Omega \setminus \{P\}$). That is*

$$f, g \in C^1(\Omega \setminus \{P\}) \cap C(\Omega)$$

Assume in addition

$$f_y(x, y) = g_x(x, y), (x, y) \neq P.$$

Then, there exists $h \in C^1(\Omega)$, so that

$$h_x = f, h_y = g,$$

including at $(x, y) = P$.

This allows us to give the anti-derivative condition as follows.

Proposition 6. *Let $\Omega \subset \mathbb{C}$ is a simply connected open set, $P \in \Omega$. Suppose that F is holomorphic in $\Omega \setminus \{P\}$ and continuous on Ω. Then, there exists H, holomorphic on U, so that $H'(z) = F$.*

2 Cauchy theorem, Cauchy integral formula

We have covered the Cauchy theorem and Cauchy integral formula. These deal with complex integrals, so we will use the notation $\oint$, whenever we integrate over closed curves. Also, if nothing else is specified, we will assume by default that the integration is over closed curves with positive (i.e. counterclockwise) orientation[1].

Theorem 4. *(Cauchy theorem) Let Ω be simply connected domain and f is holomorphic on Ω. Then, for any closed C^1 curve $\gamma : [0, 1] \to \Omega$, we have*

$$\oint_\gamma f(z)dz = 0.$$

[1] In other words, following the direction of the curve, the interior should be on your left.

An interesting corollary of the Cauchy theorem is the following.

Corollary 2. *Let U be an open subset of $\mathbb{C}$. Let γ_1, γ_2 be two curves, with the same index, so that $Int(\gamma_1) \subset Int(\gamma_2)$. Assume that f is holomorphic in $Int(\gamma_2) \setminus Int(\gamma_1)$. Then,*

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz.$$

In other words, whenever we integrate a holomorphic function on a given curve, and we can continuously deform the given curve to another one, without encountering singularities of f, the value of the integral remains the same.

For the Cauchy integral formula, one prepares by evaluating the following complex integral

$$\oint_{|\xi - z_0| = r} \frac{1}{\xi - z} = 2\pi i$$

whenever $z : |z - z_0| < r$.

Theorem 5. *(Cauchy integral formula) Let Ω be a simply connected domain in $\mathbb{C}$ and f is holomorphic on it. Let γ be a C^1 closed curve in Ω, which winds around once around $z \in \Omega$. Then,*

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{\xi - z} d\xi.$$

The applications of the Cauchy theorem and the Cauchy integral formula are nothing short of spectacular. We are going through some this week.

Theorem 6. *Let U be an open subset of $\mathbb{C}$ and $f \in H(U)$. Then, $f \in C^\infty(U)$ and for every integer k and for every curve γ with index one around $z \in U$*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi$$

Corollary 3. *Let U be an open subset of $\mathbb{C}$ and $f \in H(U)$. Then, $f \in C^\infty(U)$.*

Another corollary is

Corollary 4. *Let U be an open subset of $\mathbb{C}$ and $f \in H(U)$. Then, $f', f'', f''', \ldots$ are all holomorphic on U.*

In the same spirit, one can in fact define "analytic extensions".

Proposition 7. *Let U be an open subset of $\mathbb{C}$ and $f \in H(U)$. Let γ be a C^1 closed curve in U and $\phi \in C(\gamma)$. Introduce, for any $z \in Int(\gamma)$,*

$$f(z) := \frac{1}{2\pi i} \oint_\gamma \frac{\phi(\xi)}{\xi - z} d\xi.$$

Then, $f \in H(Int(\gamma))$.

One is then naturally left to the question for a close relation between the "boundary values" ϕ and the function inside f. The next example shows that in general there isn't much there.

Example 1. *For any $z : |z| < 1$,*

$$\int_{|\xi|=1} \frac{\bar{\xi}}{\xi - z} d\xi = 0.$$

In other words, the function $\phi(\xi) = \bar{\xi}$ generates the trivial function $f(z) = 0$ inside the unit disk.

The next result is a beautiful statement due to Morera, which says that the Cauchy theorem is essentially reversible.

Theorem 7. *(Morera) Let Ω be an open connected subset of $\mathbb{C}$ and $f \in C^0(\Omega)$. Assume that for every closed C^1 curve γ, we have*

$$\oint_\gamma f(z) dz = 0.$$

Then, f is holomorphic inside Ω.

3 Analytic functions

We then discussed complex power series. We have shown

Lemma 1. *Let $\sum_{k=0}^\infty a_k (z_0 - P)^k$ converges for a fixed point $z_0 \neq P$. Then, it converges for all $w : |w - P| < |z_0 - P|$.*

This allows one to show that the convergent set of a given power series is a ball, with radius called *radius of convergence*. This is defined as follows

$$r = \sup\{|w-P| : \sum_{k=0}^{\infty} a_k (w-P)^k \text{ converges}\}.$$

We have the following theorem (which encompasses several results in the book).

Proposition 8. *For any power series $\sum_{k=0}^{\infty} a_k (z-P)^k$, there is a (extended) number $r : 0 \leq r \leq \infty$, so that*

1. *If $r = 0$, the series diverges for all $z \neq P$.*
2. *If $z = \infty$, then the series converges for all z.*
3. *If $0 < r < \infty$, then the series converges absolutely for all $w : |w-P| < r$ and diverges for all $w : |w-P| > r$.*

Moreover, r can be found as follows

$$r = \frac{1}{\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}}.$$

Finally, for any $\epsilon > 0$, we have that the series $\sum_{k=0}^{\infty} a_k (z-P)^k$ converges uniformly in any disc of the form $D(P, r-\epsilon)$.

Note: The convergence may not be uniform on $D(P, r)$.

Another theorem states that non-trivially convergent power series are holomorphic functions.

Theorem 8. *Let $\sum_{j=0}^{\infty} a_j (z-P)^j$ has radius of convergence $r > 0$. Then,*

$$f(z) := \sum_{j=0}^{\infty} a_j (z-P)^j,$$

is holomorphic in $D(P, r)$. Moreover, each derivative may be evaluated as follows

$$f^{(k)}(z) = \sum_{j=k}^{\infty} j(j-1)\ldots(j-k+1) a_j (z-P)^{j-k},$$

where the series representing the derivatives has the same radius of convergence r.

We have also the following uniqueness result. We have much more general result coming up later on, which requires more background material.

Proposition 9. *Suppose $\sum_{j=0}^{\infty} a_j (z-P)^j$, $\sum_{j=0}^{\infty} b_j (z-P)^j$ are two power series, which converge in $D(P,r)$, $r>0$ and in addition*

$$\sum_{j=0}^{\infty} a_j (z-P)^j = \sum_{j=0}^{\infty} b_j (z-P)^j$$

for each $z : |z-P| < r$. Then, the two series are identical, that is $a_j = b_j$, $j = 0, \ldots$

We have seen that power series with non-trivial radius of convergence define holomorphic functions.

Definition 3. *Let U be an open set. We say that a function $f : U \to \mathbb{C}$ is* **analytic**, *if for every $P \in U$, there is $r = r_P > 0$, so that*

$$f(z) = \sum_{j=0}^{\infty} a_j (z-P)^j$$

for all $z : |z-P| < r$. Then necessarily $a_j = \frac{f^{(j)}(P)}{j!}$.

So, holomorphic functions are analytic functions. The question for the converse is settled in the following theorem.

Theorem 9. *Let U be an open set and $f \in H(U)$. Let $P \in U$, $D(P,r) \subseteq U$. Then,*

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(P)}{j!} (z-P)^j$$

for all $z : |z-P| < r$. That is, holomorphic functions are analytic. The radius of convergence at each point P is at least $dist(P, \partial U)$.

Combining the last two theorems, we obtain

Corollary 5. *Let Ω is a domain in $\mathbb{C}$. Then, f is holomorphic on Ω if and only if f is analytic on Ω.*

We now discuss real analytic functions.

Definition 4. *We say that $f:(a,b)\to\mathbf{R}$ is real analytic, if for every $c\in(a,b)$, there is $\delta=\delta_c>0$, so that*

$$f(x)=\sum_{j=0}^{\infty}a_j(x-c)^j,$$

and the series converges in $(c-\delta,c+\delta)\subset(a,b)$. Clearly, in such a case $a_j=\frac{f^{(j)}(c)}{j!}$. Clearly analytic functions are $C^\infty(a,b)$, but not vice versa.

We show that real analytic functions are exactly the restrictions of holomorphic (or analytic) functions to (a,b).

Proposition 10. *Let $(a,b)\subset\mathbf{R}$. Then, f is real analytic on (a,b) if and only if there exists $\Omega\subset\mathbb{C}$, so that $(a,b)\subset\Omega$ and there is a $\tilde{f}|\in H(\Omega)$, so that $\tilde{f}|_{(a,b)}=f$.*

The next topic is about the Cauchy estimates and its corollaries.

Theorem 10. *Let U be an open set and $f\in H(U)$. Let $P\in U$, $\overline{D(P,r)}\subseteq U$. Set $M=\sup_{z\in\overline{D(P,r)}}|f(z)|$. Then, for each $k=1,2,\dots$*

$$|f^{(k)}(P)|\le\frac{Mk!}{r^k}.$$

Definition 5. *We say that a function $f:\mathbb{C}\to\mathbb{C}$ is entire, if it is holomorphic on the whole $\mathbb{C}$.*

Theorem 11. *(Liouville) An entire bounded function is a constant.*

In fact, assume that

$$|f(z)|\le C(1+|z|)^k,$$

for some integer k and f is entire. Then, f is a polynomial of degree at most k.

Note: These types of results are usually applied in the negative - "An entire function cannot have polynomial growth at infinity, unless it is a polynomial". That is, non-trivial (i.e. non-polynomial) entire function grows faster than any polynomial at infinity.

This week, we deal with uniform limits of holomorphic functions. The main notion is uniform convergence on the compact subsets.

Definition 6. *Let U be an open subset of $\mathbb{C}$. We say that a sequence of functions $f : U \to \mathbb{C}$ converges uniformly on the compact subsets of U to f, denoted $f_j \rightrightarrows_{comp} f$, if for every $K \subset U$, K compact, and for every $\epsilon > 0$, there is $N = N(\epsilon, K)$, so that for all $n > N$*

$$\sup_{z \in K} |f_n(z) - f(z)| < \epsilon.$$

Equivalently, there is the Cauchy formulation, which states that uniform convergence over the compact subsets holds if for every $K \subset U$, K compact, and for every $\epsilon > 0$, there is $N = N(\epsilon, K)$, so that for all $n > m > N$

$$\sup_{z \in K} |f_n(z) - f_m(z)| < \epsilon.$$

The main theorem states that set of holomorphic functions on U is closed set under the operation "uniform convergence over the compact subsets of U".

Theorem 12. *Let U be an open set in $\mathbb{C}$. Let $\{f_j\}$ be a family of holomorphic functions on U, which converges uniformly over the compact subsets to f. Then, f is holomorphic on U.*

There is the following useful corollary of it.

Corollary 6. *Let U be an open set in $\mathbb{C}$. Let $\{f_j\}$ be a family of holomorphic functions on U, which converges uniformly over the compact subsets to f. Then, for each k $f_j^{(k)} \rightrightarrows f^{(k)}$.*

Next, we discussed the zeros of holomorphic functions.

4 Zeros of holomorphic functions, classification of singularities of meromorphic functions

Theorem 13. *Let U be a connected set, $f \in H(U)$. Then, the set of zeros, $Z = \{z \in U : f(z) = 0\}$ does not accumulation points inside of U, unless $f = 0$. Equivalently, assume that there is a sequence $z_j \in Z, \lim_j z_j = z_0 \in U$. Then $f = 0$ in U.*

Note: This does not exclude the possibility that for a non-trivial holomorphic function, there is an accumulation point Z_0 on ∂U. In fact, $f(z) = \sin(\frac{1}{1-z})$ has zeros $z_n = 1 - \frac{1}{n\pi}$, which accumulate to $1 \in \partial D(0,1)$.

Some corollaries of Theorem 13 are as follows.

Corollary 7. *Let U be a connected set, $f \in H(U)$, so that $f|_{D(P,r)} = 0$ for some $P \in U$. Then $f = 0$ on U.*

Corollary 8. *Let U be a connected set, $f, g \in H(U)$, so that $fg = 0$ on U. Then either $f = 0$ or $g = 0$. That is, the algebra $H(U)$ is an integral domain.*

Next topic is meromorphic functions.

Definition 7. *Let U is an open set and $P \in U$. We say that P is an isolated singularity for a function f, if $f \in H(U \setminus \{P\})$. For a function with a finitely many isolated singularities in a domain U, we say that it is* **meromorphic** *on U.*

Clearly, there are three distinct possibilities for isolated singularities.

1. f is bounded in a neighborhood of P. That is, there is $M > 0$, and $r > 0$, so that $D(P, r) \subset U$ and
$$\sup_{z \in D(P,r) \setminus \{P\}} |f(z)| \le M.$$
2. $\lim_{z \to P} |f(z)| = +\infty$
3. neither of the previous two holds.

The following theorem takes care of the first alternative.

Theorem 14. *(Riemann removability theorem) Let Ω be an open set, $P \in \Omega$ be an isolated singularity of $f \in H(\Omega \setminus \{P\})$, so that f is bounded in a neighborhood of P. Then, $\lim_{z \to P} f(z)$ exists and the function*
$$\tilde{f}(z) = \begin{cases} f(z) & z \neq P \\ \lim_{z \to P} f(z) & z = P \end{cases}$$
is a holomorphic on Ω.

In other words, f admits a holomorphic extension on the whole domain.

We have the following result about essential singularities.

Theorem 15. *(Casorati-Weierstrass) Let $f \in H(D(P, r_0) \setminus \{P\})$ and P is an essential singularity for f. Then, for each $r : 0 < r < r_0$, $f(D(P, r) \setminus \{P\})$ is dense in $\mathbb{C}$.*

Next topic is Laurent series.

Definition 8. *We say that a double infinite series $\sum_{j=-\infty}^{\infty} a_j$ is convergent, if*

$$\sum_{j=-\infty}^{-1} a_j < \infty, \quad \sum_{j=0}^{\infty} a_j < \infty$$

Laurent series are series in the form

$$\sum_{j=-\infty}^{\infty} a_j (z-P)^j.$$

Note: *Laurent series do not need to converge for any z*

We have the following general result about Laurent series.

Theorem 16. *Assume that a Laurent series $\sum_{j=-\infty}^{\infty} a_j (z-P)^j$ converges for some point $z \in \mathbb{C}$. Then, there exists unique $0 \le r_1 \le r_2 \le \infty$, so that*

1. *If $r_1 < r_2$, $\sum_{j=-\infty}^{\infty} a_j (z-P)^j$ converges for all $r_1 < |z-P| < r_2$, whence $f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j$ is holomorphic function in the annulus $D(P, r_2) \setminus \overline{D(P, r_1)}$.*
2. *The series $\sum_{j=-\infty}^{\infty} a_j (z-P)^j$ diverges for all $|z-P| < r_1$, $r_2 < |z-P|$.*
3. *For each $r_1 < s_1 < s_2 < r_2$, $\sum_{j=-\infty}^{\infty} a_j (z-P)^j$ converges uniformly in $s_1 \le |z-P| \le s_2$.*

Finally, there is the formula for the coefficients

$$a_j = \frac{1}{2\pi i} \int_{|\xi - P| = r} \frac{f(\xi)}{(\xi - P)^{j+1}} d\xi, r_1 < r < r_2. \tag{2}$$

In particular, this implies the uniqueness of the Laurent series for a given function f.

We saw that Laurent series are meromorphic functions. Conversely, we show that any meromorphic function has a Laurent series, at least in a neighborhood of the singular points. The theorem is a bit more general than that.

Theorem 17. *Let* $0 \le r_1 < r_2 \le \infty$ *and* $f \in H(D(P, r_2) \setminus \overline{D(P, r_1)})$. *Then, for* $z : r_1 < |z-P| < r_2$, *there is the representation*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j$$

where a_j *are given by* (2). *In particular, for isolated singularities,* $r_1 = 0$ *and we have convergence in the punctured neighborhood* $D(P, r) \setminus \{P\}$.

We can read the type of isolated singularities from the Laurent series.

Theorem 18. *Let* $f \in H(D(P, r) \setminus \{P\})$, *with a Laurent series* $f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j$. *Then,*

- P *is a removable singularity if and only if* $a_j = 0$ *for all* $j < 0$.
- P *is a pole if and only if there exists* $N \ge 1$, *so that* $a_j = 0$ *for all* $j < -N$, *but* $a_{-N} \ne 0$.
- P *is essential singularity if and only if there is a sequence* $k_j \to +\infty$, *so that* $a_{-k_j} \ne 0$.

5 Calculus of residues

Definition 9. *Let* $U \subset \mathbb{C}$ *be an open set and* $P \in U$. *Let* $f \in H(U \setminus \{P\})$. *If the Laurent series of* f *at* P *has the form*

$$f(z) = \sum_{j=-\infty}^{-2} a_j (z-P)^j + \frac{R}{z-P} + \sum_{k=0}^{\infty} a_k (z-P)^k,$$

we say that R *is the residue of* f *at the point* P. *We denote it* $R = Res_f(P)$.

Theorem 19. *Suppose* U *is an open simply connected subset of* $\mathbb{C}$, *let* $\{P_1, \dots, P_n\}$ *are isolated singularities of* f, $f \in H(U \setminus \{P_1, \dots, P_n\})$. *Let* $R_j = Res_f(P_j)$. *Then, for every closed curve* γ *in* U, *not passing through any* P_j, $j = 1, \dots, n$,

$$\oint_\gamma f(z) dz = \sum_{j=1}^{n} R_j \oint_\gamma \frac{1}{z-P_j} dz$$

Moreover, $\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-P_j}dz$ *is an integer, usually referred to as index (or winding number) of the curve* γ *with respect to* P.

Here is an efficient way to compute residues.

Proposition 11.

- *Let* P *be a simple pole for* f. *Then,*

$$Res_f(P) = \lim_{z\to P}(z-P)f(z).$$

- *If* P *is a pole of order* k, *then*

$$Res_f(P) = \frac{1}{(k-1)!}\Big((z-P)^k f(z)\Big)^{(k-1)}(P).$$

Next, we show a general result that counts the number of zeros for holomorphic functions. Before that,

Definition 10. *We say that a zero* z_0 *of a holomorphic function* $f \in H(D(z_0,\epsilon))$ *is of multiplicity* $k, k \in \mathbb{N}$, *if*

$$f(z) = (z-z_0)^k g(z),$$

where g *is also holomorphic in a neighborhood of* z_0 *and* $g(z_0) \neq 0$.

Theorem 20. *(Argument principle for holomorphic functions) Let* U *be an open set and* $f \in H(U)$. *Suppose that* $\overline{D(P,r)} \subset U$, *so that* $f|_{\partial D(P,r)} \neq 0$. *Then,*

$$\frac{1}{2\pi i}\oint_{|\xi-P|=r}\frac{f'(\xi)}{f(\xi)}d\xi = n_1 + \ldots + n_l,$$

where $n_1, n_2, \ldots, n_l$ *are the multiplicities of the zeros* $z_1, \ldots, z_l$ *inside* $D(P,r)$.

We can generalize this result to meromorphic functions as well.

Theorem 21. *(Argument principle for meromorphic functions)*

Let U *be an open set and* f *is meromorphic on* U. *Suppose that* $\overline{D(P,r)} \subset U$, *so that* f *has neither poles nor zeros on* $\partial D(P,r) = \{\xi : |\xi - P| = r\}$. *Then,*

$$\frac{1}{2\pi i}\oint_{|\xi-P|=r}\frac{f'(\xi)}{f(\xi)}d\xi = n_1 + \ldots + n_l - (k_1 + \ldots + k_m).$$

where $n_1, n_2, \ldots, n_l$ *are the multiplicities of the zeros* $z_1, \ldots, z_l$ *inside* $D(P,r)$ *and* $k_1, \ldots, k_m$ *are the orders of the poles* $q_1, \ldots, q_m$.

Some more zero counting. The next result is saying that if two holomorphic functions on a domain are "close", then they have the same number of zeros.

Theorem 22. *(Rouché theorem)*

Let U be an open set in $\mathbb{C}$ and $f, g : U \to \mathbb{C}$ are holomorphic. Let $P \in U, D(P,r) \subset \mathbb{C}$, so that for each $\xi : |\xi - P| = r$,

$$|f(\xi) - g(\xi)| < |f(\xi)| + |g(\xi)|. \tag{3}$$

Then,

$$\frac{1}{2\pi i} \oint_{|\xi-P|=r} \frac{f'(\xi)}{f(\xi)} d\xi = \frac{1}{2\pi i} \oint_{|\xi-P|=r} \frac{g'(\xi)}{g(\xi)} d\xi$$

That is, the number of zeros of f inside $D(P,r)$, counted with multiplicities coincides with the number of zeros of g inside $D(P,r)$.

Remark: The condition (3) is equivalent to requiring that $\frac{f(\xi)}{g(\xi)} \notin \mathbb{R}_-$.

Theorem 23. *(Hurwitz's theorem)*

Let U be an open and connected subset in $\mathbb{C}$ and $\{f_n\}_n : U \to \mathbb{C}$ are holomorphic and nowhere vanishing on U. If the sequence f_n converges on the compact subsets of U to f, then f is either identically zero or f vanishes nowhere on U.

Theorem 24. *(Maximum principle)*

Let Ω be a domain and $f \in H(\Omega)$. Assume that for some $P \in \Omega$, $|f(P)| = \sup_{z \in \Omega} |f(z)|$. Then, $f = const$.

Remark: If $f \neq const.$, then for every $Q \in \Omega$, $|f(Q)| < \sup_{z \in \Omega} |f(z)|$.

Corollary 9. *Let Ω be a domain and $f \in H(\Omega) \cap C(\bar{\Omega})$, $f \neq const.$. Then, there exists $P_0 \in \partial\Omega = \bar{\Omega} \setminus \Omega$, so that*

$$\max_{z \in \bar{\Omega}} |f(z)| = \max_{z \in \Omega} |f(z)| = |f(P_0)|.$$

Theorem 25. *(Schwartz lemma) Let f be holomorphic on the unit disc $D(0,1)$. Assume that*

1. $|f(z)| \leq 1$, $z \in D(0,1)$

2. $f(0) = 0$

Then, $|f(z)| \leq |z|$ *and* $|f'(0)| \leq 1$*. Moreover, if either* $|f(z_0)| = |z_0|$ *or* $|f'(0)| = 1$*, then there exists* $\alpha : |\alpha| = 1$*, so that* $f(z) = \alpha z$*.*

6 Infinite products and applications

Definition 11. *We say that the infinite product* $\prod_{n=1}^{\infty}(1 + a_n)$ *converges, if*

1. *Only finitely many* a_n *are equal to* -1*.*

2. *Let* N_0 *be so large that for all* $n > N_0$*,* $a_n \neq -1$*. Then, we require that*

$$\lim_{N\to\infty} \prod_{n=N_0+1}^{N} (1 + a_n)$$

converges, and **the limit is non-zero**.

Note that the convergence of $\prod_{n=1}^{\infty}(1 + a_n)$ implies $\lim_n a_n = 0$.

Proposition 12. *The series* $\sum_n |a_n|$ *converges if and only if* $\prod_{n=1}^{\infty}(1 + |a_n|)$ *converges.*

Note that the potential problems enlisted in Definition 11 do not apply in the case of the products in the form $\prod_{n=1}^{\infty}(1 + |a_n|)$. We say that the product $\prod_{n=1}^{\infty}(1 + a_n)$ *converges absolutely*, if $\prod_{n=1}^{\infty}(1 + |a_n|)$ converges.

Proposition 13. *Absolute convergence implies convergence for products. That is, if the product* $\prod_{n=1}^{\infty}(1 + |a_n|)$ *converges, then* $\prod_{n=1}^{\infty}(1 + a_n)$ *converges.*

In particular, if $\sum_n |a_n|$ *converges, then* $\prod_{n=1}^{\infty}(1 + a_n)$ *converges.*

Sometimes, this is used as follows: in order to check the convergence of $\prod_{k=1}^{\infty} a_k$, it suffices to check the convergence of $\sum_k |1 - a_k|$.

Theorem 26. *Suppose that $U \subset \mathbb{C}$ is an open set and $f_j \in H(U)$, so that $\sum_j |f_j(z)|$ converges uniformly on the compact subsets of U. More precisely, for each compact subset K in U, and for each $\epsilon > 0$, there exists $N = N(K, \epsilon)$, so that*

$$\sup_{z \in K} \sum_{j > N} |f_j(z)| < \epsilon$$

Then, the sequence of partial products $F_N(z) = \prod_{n=1}^N f_n(z)$ converges, uniformly on the compact subsets to a function $F \in H(U)$. Moreover, $F(z_0) = 0$ for some $z_0 \in U$ if and only if there exists $j_0 : f_{j_0}(z_0) = -1$ and the multiplicity of z_0 in the equation $F(z) = 0$ matches the multiplicity of z_0 in $f_{j_0}(z) = -1$.

The next result is the Weierstrass factorization theorem. We need a few lemmas.

Lemma 2. *For the elementary Weierstrass factors, $E_p(z) = (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^p}{p}}$, we have for each $z : |z| \leq 1$,*

$$|E_p(z) - 1| \leq |z|^{p+1}.$$

That is, $E_p(z)$ approximates 1 well.

Theorem 27. *Let $\{a_n\}$ be a sequence of non-zero complex numbers, without accumulation points (but they are allowed to repeat themselves). Suppose that p_n are integers, so that for all $r > 0$, there is*

$$\sum_n \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$

Then,

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

converges, uniformly on the compact subsets of $\mathbb{C}$ to an entire function F. Moreover the zeros of F are exactly the sequence a_n.

As a corollary, we can construct an entire function, with any prescribed set of zeros, as long as they do not have accumulation point.

Theorem 28. *Let $\{a_j\}$ be a sequence in $\mathbb{C}$ without accumulation points (but repetitions are allowed). Then, there exists an entire function f, so that f has exactly these zeros. In fact,* *assuming that* 0 *appears exactly* m *times, one can take*

$$f(z) = z^m \prod_{n=m+1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right).$$

This allows us to state the following factorization theorem.

Theorem 29. *(Weierstrass factorization theorem)*

Let f *be entire, which vanishes to order* m *at zero. Suppose* $\{a_n\}$ *are the other zeros of* f*, listed with their multiplicities. Then, there exists an entire function* $Q : Q(z) \neq 0$*, so that*

$$f(z) = z^m Q(z) \prod_{n=m+1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

7 Blashke products and Jensen's formula

For $a : |a| < 1$ and $z \in D(0,1)$, define a function

$$B_a(z) = \frac{z-a}{1-\bar{a}z}.$$

It is then easy to check

- $B_a \in H(D(0,1))$, $B_a(a) = 0$, $B_a(z) \neq 0$ whenever $z \neq 0$.
- $|B_a(z)| = 1$ if and only if $|z| = 1$.

Theorem 30. *Let* f *be holomorphic in a neighborhood of* $\overline{D(0,r)}$, $f(0) \neq 0$. *Let* $a_1, \ldots, a_k$ *are the zeros of* f*, listed with their multiplicities. Then*

$$\ln|f(0)| + \sum_{j=1}^{k} \ln\left(\frac{r}{|a_j|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta.$$

The next theorem gives a quantitative bounds on the behavior of the zeros of a *bounded holomorphic functions* inside $D(0,1)$. Namely, consider that there could be infinitely many zeros of such functions in $D(0,1)$, and they should be accumulating to the boundary.

Theorem 31. *Let $f \neq const.$ is a bounded holomorphic function on $D(0,1)$ and $\{a_j\}, a_j \neq 0$ are the zeros of f in $D(0,1)$. Then*

$$\sum_{j=1}^{\infty} (1 - |a_j|) < \infty. \tag{4}$$

On the other hand, there is the following result, which shows that (4) is sharp.

Theorem 32. *Let $\{a_j\} \subset D(0,1)$ with $\sum_{j=1}^{\infty}(1-|a_j|) < \infty$. Then, there exists a bounded holomorphic function on $D(0,1)$, with exactly these zeros. In fact, letting $a_1 = \ldots = a_m = 0$, $a_j \neq 0, j > m$, we can take*

$$f(z) = z^m \prod_{j=m+1}^{\infty} \frac{-\bar{a}_j}{|a_j|} B_{a_j}(z).$$

8 Prime number theorem

Denoting the set of prime numbers by $\mathscr{P}$, introduce the function

$$\pi(n) = \#\{p \in \mathscr{P} : 2 \leq p \leq n\}.$$

Use the notation $f \sim g$ for any two functions $f, g : \mathbf{R}_+ \to \mathbf{R}_+$, with $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. The prime number theorem is easy to state and hard to prove.

Theorem 33. *(Prime number theorem)*

$$\pi(n) \sim \frac{n}{\ln(n)}.$$

We approach the proof in a number of preparatory steps. Most of them are of independent interest, as they appeal to other special functions or techniques that are useful in other contexts.

8.1 The Riemann zeta function - round one

Introduce the Riemann zeta function for $z : \Re z > 1$

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Since $\frac{1}{|n^z|} = \frac{1}{n^{\Re z}}$ and since $\sum_{n=1}^{\infty} \frac{1}{n^a} < \infty$, for all $a > 1$, it is easy to see that with this definition

Proposition 14. *ζ is a holomorphic function in $\Omega = \{z : \Re z > 1\}$.*

There is also the product representation

Proposition 15. *(Euler product formula)*

For $z : \Re z > 1$, there is the following formula

$$\zeta(z) \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^z}\right) = 1$$

In particular, ζ does not vanish on $\Omega = \{z : \Re z > 1\}$.

This can be used to establish the following nontrivial estimate.

Proposition 16.

$$\sum_{p \in \mathscr{P}} \frac{1}{p} = \infty. \tag{5}$$

This not only says that the primes are an infinite set, but gives more detailed quantitative information on how many there are.

The estimate (5) is already a strong claim, even compared with the prime number theorem. In fact, *assuming the validity of the prime number theorem*, we can show that

$$\sum_{p \in \mathscr{P} : 1 < p < N} \frac{1}{p} \sim \ln(\ln(N)).$$

Indeed, it suffices to take $N = 2^n$. So, we have

$$\sum_{p \in \mathscr{P} : 1 < p < N} \frac{1}{p} = \sum_{k=1}^{n} \sum_{p \in \mathscr{P} : 2^{k-1} < p < 2^k} \frac{1}{p} \sim \sum_{k=1}^{n} \frac{\pi(2^k) - \pi(2^{k-1})}{2^k} \sim \sum_{k=1}^{n} \frac{1}{2^k} \frac{2^k}{k} \sim \ln(n) = \ln(\ln(N)).$$

8.2 The function θ

Set

$$\theta(x) = \sum_{p \in \mathscr{P} : 2 \le p \le x} \ln p.$$

Lemma 3. *The following are equivalent*

1. $\theta(x) \sim x$
2. $\pi(x) \sim \frac{x}{\ln(x)}$.

In other words, the prime number theorem follows from and it is actually equivalent to $\theta(x) \sim x$.

There is the following

Lemma 4. *If $\lim_{x\to\infty} \int_1^x \frac{\theta(t)-t}{t^2} dt$ exists, then $\theta(x) \sim x$.*

In other words, this reduces the proof of the prime number theorem to

$$\int_1^\infty \frac{\theta(t)-t}{t^2} dt < \infty. \tag{6}$$

8.3 The Φ function

We start with an useful lemma, which says that the Riemann zeta has an analytic extension beyond $\Re z > 1$.

Lemma 5. *The Riemann zeta function, originally defined on $\Re z > 1$, has an analytic extension to $\Re z > 0$, with a simple pole at 1. More precisely, the function*

$$\zeta(z) - \frac{1}{z-1}$$

can be extended analytically to $\Re z > 0$.

With this extension in mind (and uniqueness of extensions, see the first problem in Set 5), we can now state the Riemann hypothesis, arguably the most famous open math problem - it is the conjecture that all the zeros of ζ are on the vertical line $\Re z = \frac{1}{2}$.

Introduce for $z : \Re z > 1$,

$$\Phi(z) = \sum_{p\in\mathscr{P}} \frac{\ln(p)}{p^z}.$$

Clearly $\Phi \in H(\{\Re z > 1\})$, as the convergence over the compact subsets is guaranteed by the M-test. Using the Euler product formula, Proposition 15, we show

Lemma 6. *For $z : \Re z > 1$,*

$$\Phi(z) = -\frac{\zeta'(z)}{\zeta(z)} + \eta(z), \tag{7}$$

where η is a holomorphic function in $\Re z > \frac{1}{2}$.

As a consequence, the function Φ, originally defined in $\Re z > 1$, is thus extended to a meromorphic function in $\Re z > \frac{1}{2}$. More precisely,

- *Φ has a simple pole at 1 and in fact $\Phi(z) - \frac{1}{z-1}$ is holomorphic in a neighborhood of 1. Thst is, $Res_{\Phi}(1) = 1$,*
- *The other possible poles of Φ, in $\Re z > \frac{1}{2}$, are exactly at the zeros of ζ.*

Remark: Note that the Riemann hypothesis states precisely that all zeros of ζ are on $\Re z = \frac{1}{2}$, so Φ should not have poles other than 1.

Lemma 7. *The Riemann zeta function ζ does not have zeros on $\Re z = 1$.*

As a consequence of (7), *Φ has an analytic extension in a neighborhood of $\Re z = 1$, and it has only a simple pole at 1. Said otherwise, $\Phi(z) - \frac{1}{z-1}$ has an analytic extension in an open neighborhood of $\{\Re z \geq 0\}$.*

The function Φ has the following representation.

Lemma 8. *For $z : \Re z > 1$*

$$\Phi(z) = z \int_0^\infty e^{-zt} \theta(e^t) dt.$$

The quantity in (6), which gives us a sufficient condition under which the prime number theorem holds, can now be rewritten as

$$\int_0^\infty (e^{-t}\theta(e^t) - 1) dt < \infty. \tag{8}$$

8.4 The function f

Denoting $f(t) := e^{-t}\theta(e^t) - 1$, the goal is to show $\int_0^\infty f(t) dt < \infty$. The following lemma is an almost direct consequence of Lemma 7 and the representation in Lemma 8.

Lemma 9. *The function* $g(z) := \int_0^\infty f(t)e^{-zt}dt$*, which is well-defined holomorphic function in* $\Re z > 0$ *has an analytic extension in an open neighborhood of* $\{\Re z \geq 0\}$.

Remark: The statement *does not imply* that there is a $\delta > 0$, for which there is an analytic extension to the open set $\{z : \Re z > -\delta\}$. Namely, the boundary of the open set may converge as $\Im z \to \infty$ to the line $\Re z = 0$. What is true however is: for every $R > 0$, there exists $\delta = \delta_R$ (with virtually no control of δ_R), so that the function is analytically extendable in a neighborhood of the form

$$U_R = \{z : |z| < R, \Re z > -\delta_R\}.$$

Another, number-theoretic fact is

Lemma 10. *There is* C*, so that* $\theta(x) < Cx$*. As a consequence* $f : \mathbf{R}_+ \to \mathbf{R}_+$ *is a bounded function.*

It now remains to put together the fact that f is bounded and Lemma 9 with the following independently interesting result to conclude that $\int_0^\infty f(t)dt < \infty$.

Theorem 34. *(Integral theorem) Let* f *be a bounded, locally integrable function. Then, we define the Laplace transform of* f *in the right-half space as follows*

$$g(z) := \int_0^\infty f(t)e^{-zt}dt, \Re z > 0.$$

Note that g *is well-defined and holomorphic in* $\Re z > 0$. *Assuming that* g *has an extension to an open neighborhood of* $\Re z = 0$*, then* $\int_0^\infty f(t)dt < \infty$ *and* $g(0) = \int_0^\infty f(t)dt$.

References

[1] Green, Krantz, Function theory of one complex variable.