Notes Functional Analysis MATH 960/961

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1 Chapter I

1.1 Metric and Banach spaces

We start with a few definitions

Definition 1. *We say that* (X,d) *is a metric space, if* $d : X \times X \to \mathbf{R}_+$*, with the following properties*

1. $d(x,y) \geq 0$ *and* $d(x,y) = 0$ *if and only if* $x = y$
2. $d(x,y) = d(y,x)$
3. *(triangle inequality)* $d(x,z) \leq d(x,y) + d(y,z)$.

We say that $U \subset X$ *is an open set, if for every* $x \in U$*, there is an* $\epsilon > 0$*, so that* $x \in B_\epsilon(x) = \{y \in X : d y, x) < \epsilon\} \subset U$.

The collection $\tau = \{U \subset X : U - \text{open}\}$ *is called a topology on* X.

Definition 2. *We say that the sequence* $\{x_n\}_n \subset (X,d)$ *is a Cauchy sequence, if for every* $\epsilon > 0$*, there is* N*, so that whenever* $n > m > N$*, there is* $d(x_n, x_m) < \epsilon$.
Note: *Every convergent sequence is Cauchy.*

We say that the metric space (X,d) *is complete, if every Cauchy sequence is convergent.*
Note: *In order to show that a Cauchy sequence is convergent, it suffices to find a convergent subsequence.*

Definition 3. *Let X be a vector space, i.e. the operations $x+y$ and ax (where x, y are vectors and a is a scalar) a are well-defined. A function $\|\cdot\| : X \to \mathbf{R}_+$ is called a norm on X, if*

1. $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = 0$
2. $\|ax\| = |a|\|x\|$
3. *(triangle inequality)* $\|x + y\| \leq \|x\| + \|y\|$.

A vector space with a norm $(X, \|\cdot\|)$ is called a normed space. A normed space $(X, \|\cdot\|)$, which is complete in the metric $d(x, y) = \|x - y\|$ is called a Banach space.

A few examples:

- $l^p, 1 \leq p < \infty$,

$$l^p = \left\{ x = (x_n)_{n=1}^{\infty} : \|x\| := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \right\}.$$

- For a measure space $(M, d\mu)$, $L^p(M, d\mu), 1 \leq p < \infty$,

$$L^p(M, d\mu) = \left\{ f : M \to \mathbf{R} : \|f\| := \left(\int_M |f(x)|^p d\mu \right)^{\frac{1}{p}} \right\}.$$

- $L^\infty(M, d\mu)$,

$$L^\infty(M, d\mu) = \left\{ f : M \to \mathbf{R} : \|f\| := esssup\{|f(x)| : x \in M\right\}.$$

1.2 Compactness

Definition 4. *We say that $K \subset (X, d)$ is compact, if every open cover $K \subset \cup_\alpha U_\alpha$ has a finite subcover, i.e. there exists $\alpha_1, \ldots, \alpha_N$, so that $K \subset \cup_{j=1}^N U_{\alpha_j}$.*

We say that $K \subset (X, d)$ is sequentially compact, if every sequence $\{x_n\}$ in K has a convergent subsequence converging to $x \in K$. We say that K is precompact, if $\overline{K}$ is compact.

Proposition 1. *Let (X,d) be a complete metric space. Then, $K \subset (X,d)$ is compact if and only if K is sequentially compact.*

Equivalently, $K \subset (X,d)$ is pre-compact if and only if every sequence $\{x_n\}$ in K has a convergent subsequence.

In specific cases, one can characterize compactness efficiently.

1.2.1 Compacts in $\mathscr{C}(X,\mathbb{C})$

Let (X,d) be a compact metric space. Introduce the space of continuous functions on X

$$\mathscr{C}(X,\mathbb{C}) = \{f : X \to \mathbb{C} : f \text{ is continuous}, \|f\| = \sup_{x\in K} |f(x)|\}.$$

Theorem 1. *(Arzela-Ascolli)*

The subset $K \subset \mathscr{C}(X,\mathbb{C})$ is pre compact if and only if

1. *K is bounded, i.e.*
$$\sup_{f\in K} \|f\| < \infty.$$
2. *K is equi-continuous. That is, for every $\epsilon > 0$, there exists $\delta > 0$, so that for each $x, x' \in X : d(x,x') < \delta$,*
$$\sup_{f\in K} |f(x) - f(x')| < \epsilon.$$

1.2.2 Compacts in c_0, l^p spaces

Theorem 2. *$K \subset l^p, 1 \le p < \infty$ is pre-compact if and only if*

1. *K is bounded, i.e.*
$$\sup_{x\in K} \|x\|_{l^p} < \infty.$$
2. *For every $\epsilon > 0$, there exists N, so that*
$$\sup_{x\in K} \sum_{n=N}^{\infty} |x_n|^p < \epsilon.$$

$K \subset c_0$, 1 is pre-compact if and only if

1. *K is bounded, i.e.*

$$\sup_{x \in K} \|x\|_{c_0} < \infty.$$

2. *For every $\epsilon > 0$, there exists N, so that*

$$\sup_{x \in K} \sup_{n \geq N} |x_n| < \epsilon.$$

1.3 Bounded linear operators

Definition 5. *Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be normed spaces. A bounded linear operator $A : X \to Y$ is said to be bounded, if $\|Ax\|_Y \leq C\|x\|_X$.*

Proposition 2. *The space $B(X,Y) = \{A : X \to Y; A-\text{ bounded linear operator}\}$ can be made a normed space via the norm*

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}.$$

Note the inequality $\|Ax\| \leq \|A\|\|x\|$.

1.3.1 Finite dimensional spaces

We say that X is finite dimensional, if $X = span[e_1, e_2, \ldots, e_n]$ and $\{e_j\}_{j=1}^n$ is linearly independent. In such case $dim(X) := n$.

Theorem 3. *All norms on X are equivalent. That is, for any norm $\|x\|$ on X, there is a $C > 1$, so that $C^{-1}\|x\| \leq \|x\|_1 \leq C\|x\|$, where $\|\sum_{j=1}^n \lambda_j e_j\|_1 := \sum_{j=1}^n |\lambda_j|$.*

In other words, X is isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$. Thus, the compactness is the same, so the Heine-Borel theorem holds

Corollary 1. *$K \subset X$, with $dim(X) = n$ is compact if and only if*

1. *K is bounded*

2. *K is closed.*

The next theorem states that this previous result characterizes finite dimensional spaces.

Theorem 4. *Let $(X, \|\cdot\|)$ be a normed space. Then, the following are equivalent*

1. $dim(X) < \infty$
2. $B_X = \{x \in X : \|x\| \leq 1\}$ *is compact.*
3. $S_X = \{x \in X : \|x\| = 1\}$ *is compact.*

1.3.2 Quotient spaces

Let $(X, \|\cdot\|)$ be a normed space, $Y \subset X$, Y is a closed subspace. Then, define equivalence relation $x_1 \sim x_2 : x_1 - x_2 \in Y$. This introduces equivalence classes $[x] = \{\tilde{x} \in X : \tilde{x} \sim x\}$.

Lemma 1. *The quotient space $X/Y = \{[x] : x \in X\}$ is a normed space, under the norm*

$$\|[x]\| = \inf_{y \in Y} \|x + y\|.$$

Moreover, if X is a Banach space, then X/Y is Banach space as well.

1.4 Dual spaces

Definition 6. *For a normed space $(X, \|\cdot\|)$, we say that its dual space is $X^* = L(X, \mathbf{R})$. That is, X^* is the space of continuous linear functionals on X.*

Examples:

- Hilbert space H, with a dot product $\langle x, y\rangle$. Its dual space can be identified with itself[1].

[1] This is in essence the Riesz representation theorem, see below

- $L^p(M, d\mu), 1 \le p < \infty$. The dual space is $(L^p(M, d\mu))^* = L^q(M, d\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, every $\Lambda \in (L^p(M, d\mu))^*$ has the form

$$\Lambda f = \int_M f g d\mu, g \in L^q(M, d\mu), \|\Lambda\|_{(L^p)^*} = \|g\|_{L^q}$$

  In particular, $(l^p)^* = l^q, 1 \le p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1$.

- $c_0 = \{x = (x_n)_{n=1}^\infty : \lim_n x_n = 0, \|x\|_{c_0} = \sup_n |x_n|\}$. Its dual is $c_0^* = l^1$.

- $(l^1)^* = l^\infty$. Note however that $(l^\infty)^* \supsetneq l^1$.

1.5 Hilbert spaces

Definition 7. *Let H be a real vector space. If a bilinear form $\langle \cdot, \cdot \rangle : H \times H \to \mathbf{R}$ has the properties*

- $\langle x, x \rangle \ge 0$ *and* $\langle x, x \rangle = 0$, *if and only if* $x = 0$

- $\langle x, y \rangle = \langle y, x \rangle$

then we say that $\langle \cdot, \cdot \rangle$ is a dot product on H.

In this case define $\|x\| := \sqrt{\langle x, x \rangle}$. In order to check it is defining a norm on H, we need the following lemma.

Lemma 2.

- $|\langle x, y \rangle| \le \|x\| \|y\|$ *(Cauchy-Schwartz inequality)*

- $\|x + y\| \le \|x\| + \|y\|$.

Note that the Hilbert space norm satisfies the *parallelogram identity*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Theorem 5. *(Riesz representation theorem)*

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then for any bounded linear functional Λ on H, there exists an unique $y \in H$, so that $\Lambda x = \langle x, y \rangle$. Moreover, $\|\Lambda\|_{H^*} = \|y\|$. In other words, $H^* = H$.

Definition 8. *Let $S \subset H$, H - Hilbert space.* **Orthogonal complement** *is*

$$S^{\perp} = \{x \in H : \langle x, y \rangle = 0 \forall y \in S\}.$$

Note that $S^{\perp}$ is always a closed subspace of H.

Definition 9. *We say that a Banach space X is a direct sum of two subspaces $X = Y_1 \oplus Y_2$, if $Y_1 \cap Y_2 = \emptyset$ and $X \subset Y_1 + Y_2 = \{y_1 + y_2 : y_j \in Y_j, j = 1, 2\}$.*
Note that in such case, for every $x \in X$, there is unique pair $y_1 \in Y_1, y_2 \in Y_2$, so that $x = y_1 + y_2$.

Proposition 3. *Let H be a Hilbert space and $E \subset H$ is a subspace of it. Then*

$$H = E \oplus E^{\perp}.$$

1.6 Baire category theorem

Definition 10. *Let (X, d) be a complete metric space. Then,*

- *A is called* **nowhere dense**, *if $Int(\bar{A}) = \emptyset$.*
- *A is called* **meager**, *if $A \subset \cup_{j=1}^{\infty} A_j$, where A_j are nowhere dense.*
- *Ω is called* **residual**, *if Ω^c is meager.*

Note that if A is meager, then $\bar{A}$ is also meager. Ω is residual, if $\Omega \supset \cap_{j=1}^{\infty} U_j$, where U_j are open and dense. These are called G_{δ} sets.

Theorem 6. *(Baire category theorem)*

Let (X, d) be a metric space. Let $\{U_j\}_{j=1}^{\infty}$ be a family of open and dense sets. Then $\cap_{j=1}^{\infty} U_j$ is a dense set in X.

Corollary 2. *Let (X, d) be a metric space. Then,*

- *If Ω is residual, then Ω is dense.*
- *$X \neq \cup_{j=1}^{\infty} F_j$, where F_j is nowhere dense. Moreover, $Int(\cup_{j=1}^{\infty} F_j) = \emptyset$.*

The way this is used in practice is as follows: If $X = \cup_{j=1}^{\infty} F_j$ and F_j are closed, then there exists j_0, so that $Int(F_{j_0}) \neq \emptyset$.

2 Chapter II

2.1 Uniform boundedness principle

Theorem 7. *(UBP)*

Let $(X, \|\cdot\|)$ *and* $(Y_i, \|\cdot\|_i)$ *are Banach spaces,* $i \in I$*. Suppose that* $A_i : X \to Y_i$ *are bounded linear operators. Suppose that for every* $x \in X$*, the orbit* $\{A_i x\}$ *is bounded. That is* $\sup_i \|A_i x\| < \infty$*. Then,*

$$\sup_i \|A_i\| < \infty.$$

Corollary 3. *Suppose that* $(X, \|\cdot\|)$ *and* $(Y_i, \|\cdot\|_i), i \in I$ *are Banach spaces. Suppose that* $\sup_i \|A_i\| = \infty$*. Then, there exists* $x \in X$*, so that* $\sup_{i \in I} \|A_i x\| = \infty$*.*

Another result, which is very useful in the application is the Banach-Steinhaus theorem.

Theorem 8. *(Banach-Steinhaus)*

Let X, Y *are Banach spaces and* $A_n : X \to Y$ *is a sequence of bounded linear operators, so that* $\lim_n A_n x$ *exists for every* x*. Then*

- $\sup_n \|A_n\| < \infty$
- $AX := \lim_n A_n x$ *is a bounded linear operator*

2.2 Open mapping theorem

Definition 11. *We say that a mapping* $f : X \to Y$ *is open, if for every open set* $U \subset X$*,* $f(U)$ *is open.*

Theorem 9. *(Open mapping theorem)*

Let X, Y *are Banach spaces and* $T : X \to Y$ *is a bounded linear operator, which is onto, i.e.* $T(X) = Y$*. Then,* T *is an open mapping.*

One of the main applications is the inverse operator theorem.

Theorem 10. *Let X, Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, which is a bijection, i.e. one-to-one and onto Then, the algebraically defined inverse operator $T^{-1} : Y \to X$ is bounded.*

Some corollaries are as follows.

Corollary 4. *Suppose that X is a Banach space, so that $X = X_1 \oplus X_2$. Then, there exists a constant c, so that for every $x = x_1 + x_2$,*

$$\|x_1\| + \|x_2\| \leq c\|x_1 + x_2\|.$$

2.3 Hahn-Banach theorem

Definition 12. *Let X be a real vector space. We say that $p : X \to \mathbf{R}$ is a quasi semi-norm, if*

1. *For each $\lambda > 0$, $p(\lambda x) = \lambda p(x)$*
2. *$p(x + y) \leq p(x) + p(y)$.*

If we have $p(\lambda x) = |\lambda| p(x)$ for each $\lambda \in \mathbf{R}$, we say that p is a semi-norm.

Examples:

1. $p(x) = \|x\|$
2. For a convex set $K \subset X$, define its Minkowski functional

$$p_K(x) = \inf\{a > 0 : \frac{x}{a} \in K\}.$$

3. On l^∞, $p(x) = \limsup_n x_n$.

Theorem 11. *(Hahn-Banach theorem)*

Let X be a real normed space and $p : X \to \mathbf{R}$ be a quasi semi-norm on it. Let $Y \subset X$ be a linear subspace and $\phi : Y \to \mathbf{R}$ is a linear functional, so that $\phi(y) \leq p(y)$. Then, there exists an extension $\Phi : X \to \mathbf{R}$, so that $\Phi|_Y = \phi$ and $\Phi(x) \leq p(x), x \in X$.

Corollary 5. *(Complex version)*

Let X be a complex Banach space and $Y \subset X$ be a linear subspace. Let $\psi : Y \to \mathbb{C}$ be a complex linear functional, so that $|\psi(x)| \leq c\|x\|$. Then, there exists an extension $\Psi : X \to \mathbb{C}$, so that $\Psi|_Y = \psi$ and $|\Psi(x)| \leq c\|x\|, x \in X$.

Lemma 3. *(Properties of the Minkowksi functional)*

Let X be real topological vector space and $K \subset X$ is a convex set, with $0 \in Int(K)$. Then, its Minkowksi functional

$$p_K(x) = \inf\{a > 0 : \frac{x}{a} \in K\}$$

satisfies $p(\lambda x) = \lambda p(x)$ for each $\lambda > 0$, $p_K(x + y) \leq p_K(x) + p_K(y)$.

2.4 Applications of Hahn-Banach theorem

2.4.1 Separation of convex sets

Theorem 12. *(Hyperplane separation theorem) Let X be a real topological vector space and $K \subset X$ be convex and open subset. Then, for every $y \notin K$, there exists a linear functional $\Lambda : X \to \mathbf{R}$ and $c \in \mathbf{R}$, so that*

$$\sup_{x \in K} \Lambda(x) \leq c = \Lambda(y).$$

Theorem 13. *(Separation of two convex sets)*

Let X be a real topological vector space and $A, B \subset X$ are convex subsets, so that $Int(A) \neq \emptyset$ and $A \cap B = \emptyset$. Then, there exists a linear functional $\Lambda : X \to \mathbf{R}$ and $c \in \mathbf{R}$, so that

$$\sup_{x \in A} \Lambda(x) \leq c \leq \inf_{y \in B} \Lambda(y)$$

2.4.2 Closure of a linear subspace

For any normed space X and its dual X^*, introduce the notation

$$\langle x, x^* \rangle := x^*(x),$$

to denote the action of x^* on x. Clearly, this allows us to view $x \in X \subset X^{**}$, via the formula $x(x^*) = \langle x, x^* \rangle$.

Definition 13. *For any X real normed vector space and any set $S \subset X$, define its annihilator*

$$S^{\perp} = \{x^* \in X^* : \langle s, x^* \rangle = 0, \ \ \forall s \in S\}.$$

Note: *$S^{\perp}$ is a always a closed linear subspace of X^*.*

Theorem 14. *Let X be a Banach space, $Y \subset X$ be a linear subspace, so that $x_0 \in X \setminus \bar{Y}$. Then, $\delta = dist(x_0, Y) > 0$ and there exists $x^* \in Y^{\perp}$, so that $\|x^*\| = 1$, $x^*(x_0) = \delta$.*

Corollary 6. *For every $x \in X$, there is an $x^* \in X^* : \|x^*\| = 1, x^*(x) = \|x\|$.*

Corollary 7. *Let X be a Banach space, $Y \subset X$ be a linear subspace,Then,*

$$x \in \bar{Y} \Longleftrightarrow \langle x, x^* \rangle = 0, \forall x^* \in Y^{\perp}.$$

One might introduce for every $S \subset X^*$, $S^{\top} = \{x \in X : \langle x, s \rangle = 0, \ \ \forall s \in S\}$. In this case, Corollary 7 reads

$$\bar{Y} = (Y^{\perp})^{\top}.$$

3 Weak and Weak$*$ topology

Let X be a real vector space and $\mathscr{F}$ be a collection of real-valued linear functionals. Define the sets

$$\mathcal{V}_{\mathscr{F}} = \{\cap_{i=1}^{m} f_i^{-1}(a_i, b_i) : f_i \in \mathscr{F}, a_i < b_i\}.$$

Lemma 4. *The set*

$$\mathcal{U}_{\mathscr{F}} = \{U \subset X : \forall x \in U, \exists V \in \mathcal{V}_{\mathscr{F}}, x \in V \subseteq U\}.$$

is a topology on X. In other words, $\mathcal{V}_{\mathscr{F}}$ is a base for $\mathcal{U}_{\mathscr{F}}$, while the sets $\{f^{-1}(a, b) : f \in \mathscr{F}, a < b\}$ are a sub-base for $\mathcal{U}_{\mathscr{F}}$.

The topology $(X, \mathcal{U}_{\mathscr{F}})$ may be equivalently defined by saying that $x_\alpha \to x$ exactly when $f(x_\alpha) \to f(x)$ for each $f \in \mathscr{F}$.

3.1 Weak topology on X

Definition 14. *(Weak topology on X)*

Let X real normed vector space and $\mathscr{F} = X^*$. The corresponding topology $(X, \mathscr{U}_{X^*})$ is called **weak topology** *on X.*

Equivalently, x_α tends to x weakly (denoted $x_\alpha \rightharpoonup x$), if for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x)$.

Proposition 4. *Weak topology is weaker than the strong topology, i.e. $\mathscr{U}_{X^*} \subset \mathscr{U}_{\|\cdot\|}$. That is, every weakly open set is strongly open. Equivalently,* **every strongly convergent sequence is weakly convergent**, *i.e.*

$$\|x_\alpha - x\| \to 0 \Longrightarrow x_\alpha \rightharpoonup x.$$

Remarks: The generic converse is false.

- In $l^p, 1 < p < \infty$, $e_n \rightharpoonup 0$, while $\|e_n\| = 1$ (Exercise)
- In $C[0,1]$, $f_n \rightharpoonup f$, if and only if $\sup_n \|f_n\|_{C[0,1]} < \infty$ and f_n tends to f point-wise. (Exercise)
- In $l^p, 1 < p < \infty$, $x^n \rightharpoonup x$ if and only if $\sup_n \|x^n\|_{l^p} < \infty$ and $x^n_k \to x_k$ for each k. (Exercise)

Proposition 5. *If $\mathscr{U}_{X^*} = \mathscr{U}_{\|\cdot\|}$, then $dim(X) < \infty$.*

For example $S = \{x \in X : \|x\| = 1\}$ is always norm closed, if $dim(X) = \infty$, one has that S is not weakly closed, since 0 is in the weak closure of S. That is, if $dim(X) = \infty$, there is $x_\alpha : \|x_\alpha\| = 1$, so that $x_\alpha \rightharpoonup 0$.

The next result makes Proposition 5 ever more puzzling.

Theorem 15. *(Shur's theorem)*

In l^1, $\lim_n \|x^n - x\|_{l^1} = 0$ if and only if $x_n \rightharpoonup x$.

Note: Why is this not a contradiction with Proposition 5?

3.2 Weak$*$ topology on X^*

Definition 15. *(Weak$*$ topology on X^*)*

Let X real normed vector space, consider X^* and $\mathscr{F} = X$. The corresponding topology $(X^*, \mathscr{U}_X)$ is called **weak$*$ topology** *on X^*.*

Equivalently, x^*_α tends to x^* weak$*$ (denoted $x^*_\alpha \rightharpoonup x^*$), if for all $x \in X$, $\langle x^*_\alpha, x\rangle \to \langle x^*, x\rangle$.

Remark: If X is reflexive[2], then weak and weak$*$ topologies on X^* coincide.

Proposition 6. *On X^*, the weak$*$ topology, $(X^*, \mathscr{U}_X)$ is weaker than the weak topology $(X^*, \mathscr{U}_{X^{**}})$, which is weaker than the norm topology.*

Proposition 7. *Let X be a normed space and $K \subseteq X$ is a convex subset. Then K is closed if and only if K is weakly closed.*

Lemma 5. *(Mazur's lemma)*

Let X be a normed space and $x_n \rightharpoonup x$. Then, $x \in \overline{conv\{x_n\}}$. Equivalently, for all $\epsilon > 0$, there exists $N, \lambda_1 \geq 0, \ldots, \lambda_N \geq 0 : \sum_{j=1}^N \lambda_j = 1$, so that $\|x - \sum_{j=1}^N \lambda_j x_j\|_X < \epsilon$.

Lemma 6. *Let $x_n \rightharpoonup x$ or $x_n \rightharpoonup x$. Then, $\sup_n \|x_n\| < \infty$ and*

$$\|x\| \leq \liminf_n \|x_n\|.$$

For Hilbert spaces or more generally locally convex spaces, there is the partial reverse as follows.

Proposition 8. *Let X - Hilbert space or more generally locally convex space[3]. The following are equivalent.*

1. $x_n \rightharpoonup x$ *and* $\lim_n \|x_n\| = \|x\|$.
2. $\lim_n \|x_n - x\|_X = 0$.

[2] which happens exactly when X^* is reflexive

[3] It turns out that these are always reflexive so the weak and weak$*$ topologies coincide

3.3 Banach-Alaoglu's theorem

Here is a version of the Banach-Alaoglu's theorem, which is particularly useful in the applications.

Theorem 16. *Assume that X is a separable Banach space. Then, every bounded sequence in X^* has a weak$*$ convergent subsequence. In other words, every bounded set is weak$*$ pre-compact*[4].

The full theorem is as follows.

Theorem 17. *(Banach-Alaoglu's theorem)*

The unit ball B_{X^*} is weak$*$ compact. In other words, $(B_{X^*}, \mathscr{U}_X)$ is compact.

Remark: The theorem fails for the weak topology, unless the weak$*$ topology coincides with the weak topology. In fact $(B_{X^*}, \mathscr{U}_{X^{**}})$ is compact if and only if X is reflexive if and only if X^* is reflexive.

4 Fredholm theory

Definition 16. *Let X, Y be normed spaces and $A : X \to Y$ be a bounded linear operator. Define $A^* : Y^* \to X^*$ by the assignment*

$$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$$

We use the

Lemma 7. *Let $A : X \to Y$ is a bounded linear operator. Then, $A^* \in B(Y^*, X^*)$ and $\|A^*\| = \|A\|$.*

Lemma 8. *Let $A : X \to Y$, $B : Y \to Z$. Then $(BA)^* = A^* B^*$ and $I^* = I$.*

[4]Technically speaking, the claim is about weak$*$ sequential pre-compactness. In view of the assumption that X is separable, these two notions are the same.

Examples:

1. $A \in M_{n\times m}$, $A:\mathbf{R}^m \to \mathbf{R}^n$, $A=(a_{ij})_{1\le i\le n, 1\le j\le m}$,
$$(Ax)_i = \sum_{j=1}^{m} a_{ij}x_j.$$
$A^t = (a_{iji})_{1\le i\le n, 1\le j\le m}$, $A^t:\mathbf{R}^n \to \mathbf{R}^m$.

2. $A: l^2 \to l^2$, $(Ax)_n = x_{n+1}, n=1,\ldots$.
Then, $(A^*y)_1 = 0$, $(A^*y)_n = y_{n-1}, n=2,\ldots$.

Theorem 18. *Let $A: X \to Y$. Then,*

1. $Im(A)^{\perp} = Ker(A^*)$
2. ${}^{\perp}Im(A^*) = Ker(A)$
3. *A has dense range if and only if A^* is injective.*

Note that if $Im(A)$ is closed, then $Im(A) = {}^{\perp}Im(A)^{\perp} = Ker(A^*)^{\perp}$.

Example: $A: l^2 \to l^2$, $(Ax)_n = \frac{x_n}{n}$ is bounded operator, with dense image, but $Im(A) \subsetneq l^2$.

Lemma 9. *Let $A: X \to Y$, $x^* \in X^*$. Then, the following are equivalent (TFAE)*

1. $x^* \in Im(A^*)$
2. *There exists $c > 0$, so that for each $x \in X$,*
$$|\langle x^*, x\rangle| \le c\|Ax\|_Y.$$

4.1 Algebraic factorization of maps through invertible maps

Let X, Y be vector spaces and $A: X \to Y$ be linear map. Not all maps are (algebraically) invertible, in fact A must be injective and surjective in order to be invertible.

Introduce $\pi : X \to X_0 := X/Ker(A)$, defined $\pi(x) = [x]$ (where $[x_1] = [x_2]$ if $x_1 - x_2 \in Ker(A)$). Then, one can define $A_0 : X_0 \to Y$.

$$A_0([x]) := Ax.$$

Clearly, this definition is independent on the representative. Also, $A_0 : X_0 \to Y_0 := Im(A)$. Such a map is clearly invertible (A_0 has $Ker(A_0) = \{0\}$, and $Im(A_0) = Im(A) = Y_0$). Furthermore, the inclusion $Y_0 = Im(A) \subseteq Y$ is denoted by i. So, one can factorize $A = i \circ A_0 \circ \pi$.

Question: When is such a map A_0 continuous? When is its inverse continuous? What does it mean in terms of estimates?

The following proposition provides the answers.

Proposition 9. *Let X, Y be vector spaces and $A : X \to Y$ be linear map. Then, one has the factorization*

$$A = i \circ A_0 \circ \pi.$$

where $A_0 : X/Ker(A) \to Im(A)$ is invertible.

Suppose now that X, Y are normed vector spaces. If A is bounded, then $A_0 : X_0 \to Y_0$ is bounded as well and

$$\|A_0\|_{B(X_0,Y_0)} = \sup_{x \in X} \frac{\|A(x)\|_Y}{\inf_{\xi \in Ker(A)} \|x + \xi\|_X} \le \|A\|_{B(X,Y)}.$$

Theorem 19. *(Closed Image Theorem)*

Let $A : X \to Y$ be bounded linear map, $A^* : Y^* \to X^*$. TFAE

1. $Im(A) = {}^{\perp} Ker(A^*)$.

2. $Im(A)$ *is closed.*

3. *there exists $c > 0$, so that for all $x \in X$,*

$$\inf_{\xi \in Ker(A)} \|x + \xi\|_X \le c\|Ax\|_Y. \tag{1}$$

4. $Im(A^*) = Ker(A)^{\perp}$

5. $Im(A^*)$ *is closed.*

6. *there exists* $c > 0$*, so that for all* $x^* \in X^*$,

$$\inf_{\xi^* \in Ker(A^*)} \|x^* + \xi^*\|_{X^*} \leq c\|A^* x\|_{Y^*}.$$

Remark: These are all equivalent to $A_0 : X_0 \to Y_0$ has bounded inverse.

4.2 Some important corollaries from the Closed Image Theorem

Proposition 10. *(see Corollary 4.1.17/page 172)*

Let $A : X \to Y$, X, Y*-Banach. Then*

- A *is surjective if and only if* A^* *is injective and* $Im(A^*)$ *is closed. Equivalently,*

$$\|y^*\|_{Y^*} \leq c\|A^* y^*\|_{X^*}. \tag{2}$$

- A^* *is surjective if and only if* A *is injective and* $Im(A)$ *is closed. Equivalently,*

$$\|x\|_X \leq c\|Ax\|_Y. \tag{3}$$

A simple consequence is

Corollary 8. *Let* $A : X \to Y$ *be a bounded linear operator. Then,* A *is a bijection if and only if* A^* *is a bijection.*

4.3 Compact operators

Definition 17. *We say that* $K : X \to Y$ *is* **compact operator***, if* $K(B_X)$ *is precompact. Equivalently, for every bounded sequence* x_n, $\{Kx_n\}$ *has a convergent subsequence. In particular,* K *is bounded.*

A bounded operator $A : X \to Y$ *is called a* **finite rank operator***, if* $dim(Im(A)) < \infty$*. Note that each finite rank operator is compact.*

Lemma 10. *(Lemma 4.2.3/page 174)*

Let $K : X \to Y$ be compact. Then, if $x_n \rightharpoonup x$, then $\lim_n \|Kx_n - Kx\| = 0$.

Exercise: Show that A is finite rank if and only if there exists $x_1^*, \dots, x_N^* \in X^*$ and $y_1, \dots, y_N \in Y$, so that $A = \sum_{j=1}^N x_j^* \otimes y_j$ or

$$Ax = \sum_{j=1}^N \langle x_j^*, x \rangle y_j.$$

Also, check the exercises/examples on page 175.

Theorem 20. *(Theorem 4.2.10/page 175) We have the following.*

1. *Let $A : X \to Y$ and $B : Y \to Z$ be both bounded. If one of them is compact, then $BA : X \to Z$ is compact as well.*
2. *$K_n : X \to Y$ are compact and* $\lim_n \|K_n - K\| = 0$. *Then, K is compact as well.*
3. *K is compact if and only if K^* is compact.*

4.4 Fredholm operators

Definition 18. *Let $A : X \to Y$ be a bounded linear operator. As usual*

$$Ker(A) = \{x : Ax = 0\} \subset X; Im(A) = \{Ax : x \in X\} \subseteq Y, coKer(A) := Y/Im(A).$$

If $Im(A)$ is a closed subspace, then $coKer(A)$ is a Banach space.

A is Fredholm, if $dim(Ker(A)) < \infty, dim(CoKer(A)) < \infty$. In this case, we introduce its index

$$ind(A) = dim(Ker(A)) - dim(CoKer(A)).$$

Examples:

1. If $dim(X), dim(Y) < \infty$, then any $A : X \to Y$ is Fredholm and

   $$ind(A) = dim(X) - dim(Y)$$

   For $X = Y = \mathbf{R}^n$ and $A \in M_{n,n}$, this says $ind(A) = 0$. This is the rank-nullity theorem (i.e. $dim(Ker(A)) + dim(Im(A)) = n$) in disguise. Why?

2. if $A : X \to Y$ is a bijection, then $ind(A) = 0$.

Next is the duality theorem, which states that Fredholmness if preserved under taking adjoints.

Theorem 21. *Let $A : X \to Y$ is bounded operator. Then,*

- *If A, A^* have closed images, then*

$$dim(Ker(A^*)) = dim(CoKer(A)), dim(CoKer(A^*)) = dim(Ker(A)).$$

- *A is Fredholm if and only if A^* is Fredholm. In that case,*

$$ind(A) = -ind(A^*).$$

4.5 Some motivations

The first one is about matrices and a basic question in linear algebra. Let $A \in M_{m,n}$, so that $A : \mathbf{R}^n \to \mathbf{R}^m$, everything is real.
Question: When is the following linear equation

$$Ax = b, x \in \mathbf{R}^n, b \in \mathbf{R}^m \tag{4}$$

solvable? This is clearly solvable if and only if $b \in Im(A)$. In finite dimensions, all subspaces are closed, so we can use Theorem 20 to say $Im(A) = Ker(A^*)^{\perp}$. So, (4) is solvable if and only if $b \perp Ker(A^*)$. More specifically, we proved

Proposition 11. *The equation* (4) *is solvable if and only if $b \perp Ker(A^*)$. In particular, if $Ker(A^*) = \{0\}$, then* (4) *is solvable for each $b \in \mathbf{R}^m$.*

The other example is more sophisticated and it involves operator equations in the form

$$(Id - K)f = g, \tag{5}$$

where $K : X \to X$ is a **compact operator**, X is a Banach spaces. We will show later that such operators are Fredholm of index zero. In particular, $Im(I - K)$ is closed. So,

Proposition 12. *The equation* (5) *is solvable if and only if* $g \perp Ker(I-K^*)$*. If* $Ker(I-K^*) = \{0\}$*, then* (5) *is uniquely solvable.*

Proof. Again, $Im(I-K)$ is closed and so, $Im(I-K) = Ker(I-K^*)^{\perp}$. If $Ker(I-K^*) = \{0\}$, then $Im(I-K) = X$, whence $CoKer(I-K) = \{0\}$, so $codim(I-K) = 0$. Since $ind(I-K) = 0$, it follows that $dim(Ker(I-K)) = 0$, so $Ker(I-K) = \{0\}$. It follows that $I-K$ is a bijection and (5) is uniquely solvable. □

The results in Propositions Propositions 11 and 12, are typical to what is referred to as Fredholm alternatives.

Example: Let $K : [0,1] \times [0,1] \to \mathscr{C}$ is a continuous function. Then, the operator

$$\mathscr{K} f(x) = \int_0^1 K(x,y) f(y) dy : L^2[0,1] \to L^2[0,1],$$

is compact. Thus, an equation in the form[5]

$$\lambda f(x) - \int_0^1 K(x,y) f(y) dy = g(x), \tag{6}$$

where $\lambda \in \mathscr{C}$ and $g \in L^2[0,1]$ has an unique solution if and only if $Ker(\bar{\lambda} - \mathscr{K}^*) = \{0\}$ or

$$\bar{\lambda} z(x) = \int_0^1 \bar{K}(y,x) z(y) dy,$$

has no solutions $z \in L^2[0,1]$. Why? Note the reversed role of $(x,y) \to (y,x)$, this is not a typo.

4.6 Characterization of Fredholm operators

We start with an important lemma in the theory, which may be of independent interest.

Lemma 11. *(Lemma 4.3.9) Let* $D : X \to Y$*,* X, Y *are Banach spaces.*

Then, D has a finite dimensional kernel and closed image *if and only if* *there exists a Banach space* Z *and a compact operator* $K : X \to Z$*, so that*

$$\|x\|_X \le c(\|Dx\|_Y + \|Kx\|_Z). \tag{7}$$

[5] This is a toy version of something that appears very frequently in PDE theory!

Here is the characterization of the Fredholmness. Roughly speaking, Fredholm operators are those that are invertible modulo compacts. *Note the typo in the book, the partial inverse F, must be $F : Y \to X$ instead of $F : X \to Y$.*

Theorem 22. *(Theorem 4.3.8)*

Let $A : X \to Y$ be a bounded linear operator, X, Y are Banach spaces. Then, the following are equivalent.

1. *A is Fredholm*
2. *There exists a bounded linear operator $F : Y \to X$, so that $Id_X - FA : X \to X$, $Id_Y - AF : Y \to Y$ are both compacts.*

Comments:

1. From the proof, it is clear that it is enough to assume that there are two (maybe different ones), $F_1, F_2 : Y \to X$, so that $I_X - F_1 A$ and $Id_Y - AF_2$ are compacts. This is sometimes useful.
2. By Theorem 20, $K(X) = \{K : X \to X, K - \text{compact}\}$ is a closed, two side ideal, in the algebra $B(X)$. Thus, one may define the Calkin algebra

   $$L(X) = B(X)/K(X),$$

   In it, one may state the theorem as follows A is Fredholm if and only if $[A]$ is invertible in the Calkin algebra (with inverse $[F]$ as in the theorem).
3. For every compact operator $K : X \to X$, operators of the type $Id - K : X \to X$ is Fredholm. Just apply the theorem with $F = Id$.

Exercise: If $B : X \to X$ is invertible and $K : X \to X$ is compact, then $B - K$ is Fredholm. The main focus this week is on how to compute the index.

4.7 Composition of Fredholm operators

Theorem 23. *(Theorem 4.4.1) Let* $A : X \to Y$ *and* $B : Y \to Z$ *are both Fredholm. Then,* $BA : X \to Z$ *is Fredholm and*

$$ind(BA) = index(A) + ind(B)$$

4.8 Stability of the Freholm index

The following result is a basic result in the theory, stating that a small (in norm) perturbation does not change the index and neither does perturbation by compact (even it is a large compact operator).

Theorem 24. *(Theorem 4.4.2) Let* $D : X \to Y$ *is Fredholm. Then*

1. *If* $K : X \to Y$ *is compact, then* $D + K$ *is Fredholm and*

   $$ind(D+K) = ind(D)$$

   In other words, adding compact to a Fredholm preserves the Fredholmness and keeps the index unchanged.

2. *There exists a constant* $\epsilon > 0$*, so that whenever* $P : X \to Y$ *is a bounded operator,* $\|P\| < \epsilon$*, then* $D + P$ *is Fredholm and* $ind(D+P) = ind(D)$*.*

   In other words, the Fredholm operators are an open set in the space of bounded operators $B(X, Y)$ **and** $ind : Fredholm \to \mathbb{N}$ **is locally a constant. In fact,**

   $$Fredholms = \cup_{n\in\mathbb{Z}} \Omega_n,$$

   where each set $\Omega_n = \{A : X \to Y, ind(A) = n\}$ **is a component in the set of Fredholm operators.**

4.9 Applications

Immediately from the previous results, for each $K : X \to X$ compact, we have $ind(I+K) = ind(I) = 0$.

Theorem 25. *Let $K : X \to X$ be a compact operator. Then, for each $\lambda \in \mathbb{C}, \lambda \neq 0$,*

1. $dim(Ker(\lambda - K)) < \infty$.

2. $dim(Ker(\lambda - K)) = dim(Ker(\lambda - K^*))$

3. *Suppose* $Ker(\lambda - K) = \{0\}$. *Then,* $(\lambda Id - K)$ *is invertible.*

4. *(Fredholm alternative) The equation*

$$(\lambda - K)f = g, f, g \in X \tag{8}$$

has solutions if and only if $g \perp Ker(\lambda - K^*)$. *More specifically, If* $Ker(\lambda - K) = \{0\}$, *then* (8) *is uniquely solvable, in fact* $f = (\lambda - K)^{-1}g$.

If $Ker(\lambda - K) \neq \{0\}$, *let* $n = dim(Ker(\lambda - K)) \geq 1$. *There exist* $x_1, \ldots, x_n \in X$, *a basis of* $Ker(\lambda - K)$ *and* $x_1^*, \ldots, x_n^* \in X^*$ *a basis of* $Ker(\lambda - K^*)$, *so that* (8) *has solutions if and only if* $\langle x_j^*, g\rangle = 0$. *If this is satisfied,* $g \in Im(\lambda - K)$, $(\lambda - K) : X \to Im(\lambda - K)$ *is invertible and the general solution of* (8) *is in the form*

$$f = (\lambda - K)^{-1}g + \sum_{j=1}^{n} \mu_j x_j.$$

Remark: The condition $\lambda \neq 0$ is crucial. The theorem fails for $\lambda = 0$.

5 Spectral theory

The first part is section 5.1 in the book (pages 198-202), which introduces the concept of a Banach space over the complex numbers. I have mentioned several times how various things work in that case. Bottom line is that all the theorems remain the same, including property of norms, linear functionals, Hahn-Banach[6], open mapping theorem, closed graph theorem, inverse function theorem, Banach-Alaoglu theorem, Fredholm theory. I recommend you read these pages on your own.

[6]which needs a little extension from the real case, which I mentioned about in class

5.1 Integration

We investigate the following:

Question: How does one integrate Banach space valued (B - valued) functions? How much regularity does one need and what it means for B - valued functions?

A class that will be enough for our purposes is continuous functions, although there is a notion of Riemann/Lebesgue integrable B - valued functions.

Lemma 12. *(Integration of continuous functions, page 202) Let X be a Banach space (real or complex) and $x:[a,b]\to X$ be a continuous B - valued function. Then, there exists an unique $\xi\in X$, so that "$\xi=\int_a^b x(t)dt$". Formally, for every $x^*\in X^*$,*

$$\langle x^*,\xi\rangle=\int_a^b\langle x^*,x(t)\rangle dt.$$

The approach above is very common in how one passes from B - space valued functions to "regular" scalar valued functions. The next lemma is a prime example.

Lemma 13. *Let X be a Banach space (real or complex), $x,y:[a,b]\to X$ are continuous B - valued functions. Then,*

1. *(Linearity of the integral)*
$$\int_a^b(x(t)+cy(t))dt=\int_a^b x(t)dt+c\int_a^b y(t)dt.$$
2. *For $a<c<b$,*
$$\int_a^b x(t)dt=\int_a^c x(t)dt+\int_c^b x(t)dt.$$
3. *Let $A:X\to Y$ be bounded linear operator, then*
$$A\int_a^b x(t)dt=\int_a^b Ax(t)dt.$$
4. *(Fundamental theorem of calculus)*

   Let $x:[a,b]\to X$ be continuously differentiable, i.e. there is a continuous B - valued function $g(t)$, so that $\lim_{h\to 0}\|\frac{x(t+h)-x(t)}{h}-g(t)\|=0$ (and then $\dot{x}(t)=g(t)$). Then,
$$\int_a^b\dot{x}(t)dt=x(b)-x(a).$$

5. *(change of variables formula)*

   Let $\phi : [\alpha, \beta] \to [a, b]$ *be a differentiable and invertible transformation. Then*

$$\int_a^b \dot{x}(t)\,dt = \int_\alpha^\beta x(\phi(s))\phi'(s)\,ds.$$

6. *(Triangle inequality for integrals)*

$$\|\int_a^b x(t)\,dt\| \le \int_a^b \|x(t)\|\,dt$$

7. *If*

$$x(t) = x_0 + \int_a^t y(s)\,ds,$$

   then x *is continuously differentiable with* $\dot{x}(t) = y(t)$.

Next topic is holomorphic B valued functions.

5.2 Holomorphic functions

Definition 19. *Let* $\Omega \subset \mathbb{C}$ *be an open set,* X *is a complex Banach space,* $f : \Omega \to X$ *is continuous. We say that* f *is holomorphic, denoted* $H(\Omega)$*, if*

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \in X,$$

exists and $f' : \Omega \to X$ *is continuous function.*

Let γ *be a* C^1 *curve in* Ω*, i.e.* $\gamma : [a, b] \to \Omega$*,* $\gamma \in C^1(a, b) \cap C[a, b]$*. Then,*

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(s))\gamma'(s)\,ds.$$

It is tempting to ask:

Question: Is it true that f is B valued holomorphic function if and only if $z \to \langle x^*, f(z) \rangle$ is (scalar) holomorphic for each $x^* \in X^*$

The next lemma shows that and it is in fact more general.

Lemma 14. *Let $\Omega \subset \mathbb{C}$ be an open set, X, Y are complex Banach spaces and $A : \Omega \to L(X, Y)$ is a weakly continuous function, i.e. for each $x \in X, y^* \in Y^*$, $z \to \langle y^*, A(z)x\rangle$ is continuous function.*

Then, the following are equivalent:

1. *A is holomorphic.*
2. *For each $x \in X, y^* \in Y^*$,*
$$z \to \langle y^*, A(z)x\rangle$$
is holomorphic on Ω.
3. *(Cauchy theorem) For each $z_0 \in \Omega$ and $r_0 > 0$, so that $\overline{B_r(z_0)} \subset \Omega$,*
$$\langle y^*, A(\omega)x\rangle = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{\langle y^*, A(z)x\rangle}{z-\omega} dz, \tag{9}$$
for each $r : 0 < r < r_0$ and $\omega : |\omega - z_0| < r$.

Remarks:

- Formula (9) may be generalized to closed curves of index one around z_0.
- If you apply this to the mapping $A(z) = f(z)x^*$, where $f : \Omega \to Y$ and x^* is arbitrary non-zero element of X^*, we obtain that $f : \Omega \to Y$ is holomorphic if and only if f is weakly holomorphic, i.e. for every $y^* \in Y^*$, $z \to \langle y^*, f(z)\rangle$ is holomorphic. Also, applying (9) to this, we obtain the Cauchy formula
$$f(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-\omega} dz. \tag{10}$$

Some exercises that are good results.

Proposition 13. *Let X be a Banach space, $f : \Omega \to X$ be holomorphic. Then,*

1. *$f' : \Omega \to X$ is also holomorphic.*

2.

$$f^{(n)}(\omega) = \frac{n!}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{(z-\omega)^{n+1}}dz. \tag{11}$$

3. Let $\{a_n\}_n$ *is a sequence of elements in* X*. Suppose that* $\limsup_{n\to\infty}\|a_n\|^{\frac{1}{n}} < \infty$*, so that*

$$\rho := \frac{1}{\limsup_{n\to\infty}\|a_n\|^{\frac{1}{n}}} > 0.$$

Then, the formula

$$f(z) := \sum_{n=0}^{\infty} a_n z^n,$$

defines a B *valued function on the domain* $B_\rho(0)\subset\mathbb{C}$*. Also,*

$$a_n = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{n+1}}dz, r<\rho.$$

The function that we really want to consider is, for $A\in L(X)$, the operator valued function $z\to(z-A)^{-1}$, whenever it exists. This is called resolvent set $\rho(A)$, which happens to be open subset of $\mathbb{C}$. It is also the complement of the spectrum $\sigma(A)$.

5.3 Spectrum

Definition 20. *Let* X *be a complex Banach space and* $A\in L(X)$ *be bounded linear operator.* **The spectrum** *is introduced as follows*

$$\sigma(A) = \{\lambda\in\mathbb{C} : (\lambda - A) \text{ is not bijective/invertible}\}$$

Equivalently **the resolvent set** $\rho(A) = \mathbb{C}\setminus\sigma(A)$ *can be defined independently as*

$$\rho(A) = \{\lambda\in\mathbb{C} : (\lambda - A) \text{ is bijective/invertible}\}$$

Question: What are the reasons for $\lambda - A$ not to be invertible? It is either not injective or not surjective or both. We then subgroup them in two distinct/disjoint subgroups as follows - it is either *A*) not injective, or *B*) it is injective, but not surjective.

You can further group them in three distinct major groups(but be aware that there are other ways to group them, which makes it so confusing!) - basically group B is split in additional two groups. More specifically:

1. Non-trivial kernel - $Ker(\lambda - A) \neq \{0\}$, i.e. $\lambda - A$ is not injective.

2. $Ker(\lambda - A) = \{0\}$, but $Im(\lambda - A)$ is not dense in X.

3. $Ker(\lambda - A) = \{0\}$, $Im(\lambda - A)$ is dense in X, but $Im(\lambda - A) \neq X = \overline{Im(\lambda - A)}$.

The reasons for this split are historical and are partially motivated by the stability of these parts of the spectrum under perturbations.
Eventually $\sigma(A) = P\sigma(A) \cup R\sigma(A) \cup C\sigma(A)$, as follows.

Definition 21.

- **Point spectrum** - *$\lambda - A$ is not injective or $Ker(\lambda - A) \neq \{0\}$.*

$$P\sigma(A) = \{\lambda \in \mathbb{C} : Ker(\lambda - A) \neq \{0\}\}.$$

- **Residual spectrum** *$Ker(\lambda - A) = \{0\}$, but $Im(\lambda - A)$ is not dense in X.*

$$R\sigma(A) = \{\lambda \in \mathbb{C} : Ker(\lambda - A) = \{0\}, \overline{Im(\lambda - A)} \neq X\}.$$

- **Continuous spectrum** *$Ker(\lambda - A) = \{0\}$, $\overline{Im(\lambda - A)} = X$, but the image is not closed, or $Im(\lambda - A) \neq \overline{Im(\lambda - A)} = X$.*

$$C\sigma(A) = \{\lambda \in \mathbb{C} : Ker(\lambda - A) = \{0\}, Im(\lambda - A) \neq \overline{Im(\lambda - A)} = X\}.$$

Remarks:

- In finite dimensions $A : X \to X$, $dim(X) = n$, $\sigma(A) = P\sigma(A)$, while $R\sigma(A) = C\sigma(A) = \emptyset$. Moreover, $\sigma(A)$ is a finite set

$$\#\sigma(A) \leq n$$

This is just the fact that a matrix $\lambda - A$ is non-invertible if and only if $\det(\lambda - A) = 0$. λ is then called an eigenvalue. In this case,

$$\sigma(A) = \{\lambda : \det(\lambda - A) = 0\}, \#\sigma(A) = n,$$

if one counts eigenvalues with respective multiplicity.

- $X = l^2$. For the shift operators $A, B : l^2 \to l^2$ defined by $Ax = (x_2, x_3, \ldots), Bx = (0, x_1, \ldots)$, we have

$$\begin{aligned}\sigma(A) &= \{\lambda : |\lambda| \le 1\}, P\sigma(A) = \{\lambda : |\lambda| < 1\}, R\sigma(A) = \emptyset, C\sigma(A) = \{\lambda : |\lambda| = 1\} \\ \sigma(B) &= \{\lambda : |\lambda| \le 1\}, P\sigma(B) = \emptyset, R\sigma(B) = \{\lambda : |\lambda| < 1\}, C\sigma(B) = \{\lambda : |\lambda| = 1\}\end{aligned}$$

Remark: For A, eigenvectors are $x_\lambda = (\lambda, \lambda^2, \ldots), |\lambda| < 1$, so that $Ax_\lambda = \lambda x_\lambda$. For B, clearly $(\lambda - B)x = 0$ implies $x = 0$.

Lemma 15. *Let $A : X \to X$ be a bounded linear operator, $A^* : X^* \to X^*$ is the dual operator. Then*

- *$\sigma(A)$ is a compact subspace of $\mathbb{C}$*
- *$\sigma(A^*) = \sigma(A)$*
- $$\begin{aligned}&P\sigma(A^*) \subset P\sigma(A) \cup R\sigma(A), R\sigma(A^*) \subset P\sigma(A) \cup C\sigma(A), C\sigma(A^*) \subset C\sigma(A) \\ &P\sigma(A) \subset P\sigma(A^*) \cup R\sigma(A^*), R\sigma(A) \subset P\sigma(A^*), C\sigma(A) \subset R\sigma(A^*) \cup C\sigma(A^*).\end{aligned}$$

Proof. For the boundedness of $\sigma(A)$, we show that $\sigma(A) \subset \{\lambda : |\lambda| \le \|A\|\}$. Indeed, let $\lambda : |\lambda| > \|A\|$. Then,

$$\lambda - A = \lambda(I - \lambda^{-1}A).$$

Since $\|\lambda^{-1}A\| = \frac{\|A\|}{|\lambda|} < 1$,, the operator $(I - \lambda^{-1}A)$ is invertible by von Neumann, see Lemma 22. It remains to show that $\sigma(A)$ is closed. It is equivalent to show that $\rho(A)$ is open. This is again Lemma 22. Indeed, let $\lambda_0 \in \rho(A)$. Then, for all $\lambda : |\lambda - \lambda_0| \le \frac{1}{\|(\lambda_0 - A)^{-1}\|}$, we have by Lemma 22 that

$$\lambda - A = (\lambda - \lambda_0) + (\lambda_0 - A) = (\lambda_0 - A)(I + (\lambda - \lambda_0)(\lambda_0 - A)^{-1}).$$

Again, by von Neumann, if $\|(\lambda - \lambda_0)(\lambda_0 - A)^{-1}\| < 1$, we have invertibility of $\lambda - A$, so $\rho(A)$ is an open set.

2) follows from Corollary 8, which states that $\lambda \in \rho(A)$ if and only if $\lambda - A$ is bijection, if and only if $\lambda - A^*$ is a bijection if and only if $\lambda \in \rho(A^*)$. □

Next, we discuss the resolvent of A, whenever it is defined.

Lemma 16. *Let X be a complex Banach space and $A \in B(X)$. Then $\rho(A)$ is an open set and define, for $\lambda \in \rho(A)$,*

$$R_\lambda(A) := (\lambda - A)^{-1}$$

Then $R : \rho(A) \to B(X)$ is a holomorphic $B(X)$ valued mapping, which satisfies **the resolvent identity**

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda) R_\lambda(A) R_\mu(A), \lambda, \mu \in \rho(A). \tag{12}$$

In particular, its complex derivative satisfies

$$R'(\lambda) = -R_\lambda(A)^2.$$

Remark: Note that the resolvents commute, i.e. $R_\lambda(A)R_\mu(A) = R_\mu(A)R_\lambda(A)$, if we apply (12) with μ instead of λ and λ instead of μ.

We first establish basic properties of spectra.

5.4 Spetral radius formula

We first define spectral radius.

Definition 22. *For a bounded operator $A \in L(X)$, define its spectral radius*

$$r_A := \inf\left\{\mu > 0 : \sigma(A) \subset \{z : |z| < \mu\}\right\} = \sup_{\lambda \in \sigma(A)} |\lambda| \geq 0.$$

Theorem 26. *Let X be non-trivial complex Banach space and $A \in L(X)$. Then, $\sigma(A)$ is non-empty and*

$$r_A = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}.$$

Remark: Part of the claim is that the limit $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ exists.

5.5 Spectrum of compact operators

Most of the stuff here is already contained in Theorem 25, but some of the notions come up for more general operators. Clearly,

$$Ker((\lambda I - A)) \subseteq Ker((\lambda I - A)^2) \subseteq \ldots \subseteq Ker((\lambda I - A)^k) \subseteq Ker((\lambda I - A)^{k+1}) \subseteq \ldots$$

Here, the elements $Ker((\lambda I - A))$ are called eigenvectors, while the elements of $Ker((\lambda I - A)^2)$ are all elements, which are either eigenvectors or (first generations) adjoint eigenvectors . Think Jordan normal forms

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

so that $e_1 \in Ker(\lambda - J)$, while $e_2 \in Ker((\lambda - J)^2) \setminus Ker(\lambda - J)$ and so on. *The generalized eigenspace* is

$$E_\lambda(A) = \cup_{k=1}^\infty Ker((\lambda I - A)^k).$$

If $Ker((\lambda I - A)^k) = Ker((\lambda I - A)^{k+1})$, we say that it stabilizes and in fact $E_\lambda = Ker((\lambda I - A)^k)$. This does not have to happen for general A!

Theorem 27. *(Spectrum of compact operators) Let X be a complex Banach space and A is a compact operator on it. Then, $\sigma(A) \setminus \{0\} \subset P\sigma(A)$ and*

1. *For any $\lambda \in \sigma(K) \setminus \{0\}$, $dim(E_\lambda) < \infty$ and in fact there exists m, so that*

   $$Ker((\lambda I - A)^m) = Ker((\lambda I - A)^{m+1})$$

   In such a case, $E_\lambda(A) = Ker((\lambda I - A)^m)$ and

   $$X = E_\lambda \oplus Im((\lambda I - A)^m). \qquad (13)$$

2. *Every non-zero eigenvalue is an isolated point in $\sigma(A)$.*

5.6 Holomorphic functional calculus

We would like to define $f(A) \in L(X)$ for every holomorphic function f defined on Ω, which contains $\sigma(A)$.

Definition 23. *For every curve $\gamma \subset \Omega$, so that $\sigma(A) \subset Int(\gamma)$, we require that for every $\lambda \in \sigma(A)$,*

$$ind_\gamma(\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\lambda} = 1.$$

Then, for every $f \in H(\Omega)$, introduce

$$f(A) := \frac{1}{2\pi i}\int_\gamma f(z)(z-A)^{-1}dz$$

Remark: By the Cauchy theorem, the choice of curve γ is non-essential, as long as the condition $ind_\gamma(\lambda) = 1$ for all $\lambda \in \sigma(A)$ is satisfied.

Theorem 28. *Let X be a complex Banach space and $A \in L(X)$. Then,*

1. *Let $\Omega \subset \mathbb{C}$, so that $\sigma(A) \subset \Omega$. Then, for every $f, g \in H(\Omega)$,*

$$(f+g)(A) = f(A) + g(A), (fg)(A) = f(A)g(A).$$

2. *$p(z) = \sum_{k=0}^N a_k z^k$, then $p(A) = \sum_{k=0}^n a_k A^k$.*

3. **Spectral mapping theorem**

$$\sigma(f(A)) = f(\sigma(A)).$$

4. *$f : \Omega \to U$, $g : U \to \mathbb{C}$, f, g holomorphic, then*

$$g(f(A)) = (g \circ f)(A).$$

5. *Let $\sigma(A) = \Sigma_0 \cup \Sigma_1$, where Σ_0 and Σ_1 are disjoint. Define the holomorphic functions $f_0|_{\Sigma_0} = 1$, $f_0|_{\Sigma_1} = 0$ and $f_1 = 1 - f_0$. Then, $P_0 := f_0(A)$, $P_1 = f_1(A)$ are projections, i.e. $P_j^2 = P_j$, $j = 1, 2$, $P_0P_1 = P_1P_0 = 0$ and they induce an A invariant decomposition $X = X_0 \oplus X_1$, $X_j = P_j(X)$, so that $A_j := (zf_j)(A) = AP_j : X_j \to X_j$ and $\sigma(A_j) = \Sigma_j$.*

5.7 Adjoint operators

We now specialize on operators on a Hilbert space. It is largely a repetition of the previously studied material, but now in the case of Hilbert spaces with complex scalars. *One* *subject, which is often subject of a lot of confusion, that I would like to discuss in detail, is adjoint versus dual operators.*

For complex matrices, on $\mathbf{R}^n$, we have two operations of adjoints - A^t and A^*. Namely, if $A = (a_{ij})_{i,j=1}^n$,

$$A^t = (a_{ji})_{i,j=1}^n, \ \ A^* = (\overline{a_{ji}})_{i,j=1}^n.$$

Thus, for the duality operation, $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$, we have $\langle Ax, y\rangle = \langle x, A^t y\rangle$, whereas for the dot product $(x, y) = \sum_{j=1}^n x_j \bar{y}_j$, we have $(Ax, y) = (x, A^* y)$. Similarly for Hilbert spaces, we have dual and adjoint operators. By Riesz representation theorem, one can define the duality operation in terms of the dot product, as follows

$$\langle x, y\rangle := (x, \bar{y})$$

Definition 24. *Let $H, (\cdot,\cdot)$ be a complex Hilbert space. Recall its dot product is sesquilinear, i.e. $(a, \lambda b) = \bar{\lambda}(a, b)$. Then, given $A \in L(H)$, the adjoint operator is defined as the unique operator A^*, with*

$$(Ax, y) = (x, A^* y).$$

From now on, whenever we talk about Hilbert spaces, A^* would mean the adjoint operator, rather than the dual one (I wish the standard notation were A^t for the dual, say in the previous chapter, but that is not the case!).

Next, here are some properties of the adjoint operator A^*.

Lemma 17. *(see Lemma 5.9, page 226) Let H be a Hilbert space and $A \in L(H)$. Then,*

- $A^* \in L(H)$ *and* $\|A^*\| = \|A\|$.
- $(AB)^* = B^* A^*$, $(\lambda A)^* = \bar{\lambda} A^*$.
- $A^{**} = A$.
- $Ker(A^*) = Im(A)^\perp$, $\overline{Im(A^*)} = Ker(A)^\perp$.
-
$$\sigma(A^*) = \overline{\sigma(A)}.$$

5.8 Normal operators

Definition 25. *Let H be a complex Hilbert space and $A \in L(H)$. We say that*

- A **is normal**, *if* $AA^* = A^*A$
- A **is unitary**, *if* $AA^* = A^*A = Id$ *or* $A^* = A^{-1}$.
- A **is self-adjoint**, *if* $A^* = A$.

Note that every self-adjoint and unitary is normal as well.

Examples:

- Matrices $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} = \bar{a}_{ji}$ are self-adjoint operators on $\mathbb{C}^n$.
- $A : L^2[0,1] \to L^2[0,1]$, $Ax(t) = f(t)x(t)$, where f is periodic continuous function. $M^*x(t) = \bar{f}(t)x(t)$. So, M is always normal and it is self-adjoint if and only if f is real-valued.
- The double infinite space $l^2 = \{x = \{x_n\}_{n=-\infty}^{\infty} : \|x\| = \left(\sum_{n=-\infty}^{\infty} |x_n|^2\right)^{\frac{1}{2}}\}$ and the shift operators $(Sx)(n) = x_{n+1}$, $(Rx)_n = x_{n-1}$. Note that $S^* = R, R^* = S$ and

$$SS^* = SR = Id = RS = S^*S.$$

  Thus, S, R are unitary operators on l^2.

Exercise 1. *Let A be self-adjoint. Then $A = 0$ if and only if $(Ax, x) = 0$ for all x.*

Proposition 14. *Let H be a complex Hilbert space and $A \in L(H)$. Then,*

- *A is normal if and only if $\|A^*x\| = \|Ax\|$ for all $x \in H$.*
- *A is unitary, if $\|Ax\| = \|A^*x\| = \|x\|$.*
- *A is self-adjoint, if $(Ax, x) \in \mathbf{R}$ for all $x \in H$.*

Theorem 29. *(Spectrum of normal operators) Let H be a complex Hilbert space and $A \in L(H)$ be a normal operator. Then,*

1. $\|A^n\| = \|A\|^n$ *for every* $n \in \mathbb{N}$.

2. $\|A\| = \sup_{\lambda \in \sigma(A)} |\lambda|$.

3. $R\sigma(A^*) = R\sigma(A) = \emptyset$, $P\sigma(A^*) = \{\bar{\lambda} : \lambda \in P\sigma(A)\}$.

4. *If A is unitary, then* $\sigma(A) \subset \{z : |z| = 1\}$.

Proof. 1) is easy, see page 229. 2) follows from it by the spectral radius formula. Part 3) is also well done there. □

Lemma 18. *Let $A = A^*$ on a Hilbert space and $\lambda \neq \mu \in P\sigma(A)$, with corresponding eigenvectors x, y, i.e. $Ax = \lambda x$, $Ay = \mu y$. Then, $(x, y) = 0$.*

Theorem 30. *(Characterization of normal compact operators) Let H be a complex Hilbert space and $A \in L(H)$ be a compact and normal operator. Then, there exists an orthonormal sequence $\{e_n\}_{n\in I}$ and $\lambda_i, i \in I$, so that $\lim_i \lambda_i = 0$ and*

$$Ax = \sum_{i\in I} \lambda_i (x, e_i) e_i.$$

5.9 Spectrum of a self-adjoint operators

Theorem 31. *Let H be a complex Hilbert space and $A \in L(H)$ be a self-adjoint operator. Then,*

1. $\sigma(A) \subset \mathbf{R}$

2. $\sup \sigma(A) = \sup_{\|x\|=1} (Ax, x)$

3. $\inf \sigma(A) = \inf_{\|x\|=1} (Ax, x)$.

4. $\|A\| = \sup_{x:\|x\|=1} |(Ax, x)|$.

We would like to extend the functional calculus, in the case of a self-adjoint operator, to the algebra of continuous functions. The idea is to construct, for a fixed self-adjoint operator A on a Hilbert space H, an algebra homomorphism $\Phi_A : C(\sigma(A)) \to L(H)$, so that $\Phi_A(e_0) = A$, where $e_0(x) = x$.

5.10 Banach algebras

Definition 26. *We say that a Banach space A is an unital Banach algebra, if in addition to the operations addition and multiplication by scalar, there are the operations product $(a,b) \to ab$ (and the unit element $\mathbb{1} : a\mathbb{1} = \mathbb{1}a = a$) and the star operation $a \to a^*$, with the properties*

$$\|ab\| \le \|a\|\|b\|,\ (ab)^* = b^* a^*,\ \mathbb{1}^* = \mathbb{1},\ (\lambda a)^* = \bar{\lambda} a^*,\ a^{**} = a,$$

and the star property

$$\|a^* a\| = \|a\|^2.$$

A is called commutative, if $ab = ba$ for all $a, b \in A$.

Examples:

- $C(K)$ - the continuous functions on a compact space K is commutative C^* algebra.
- $L(H)$, with the regular $*$ operation of adjoints. This is highly non-commutative Banach algebra.
- the space $l^1(\mathbb{Z}) = \{(x_n)_{n=-\infty^\infty} : \|x\| = \sum_{n=-\infty}^{\infty} |x_n|\}$, with the regular operations and a product operation given by $z = x.y$

  $$z_k = \sum_{n=-\infty}^{\infty} x_{k-n} y_n.$$

  is a commutative Banach algebra (Check!).

5.11 Continuous functional calculus for self-adjoint operators

The next lemma begins to build the homomorphism Φ_A, namely with the polynomials.

Definition 27. *Let A, B be C^* algebras, and $\Phi : A \to B$. We say that Φ is C^* algebra homomorphism, if*

$$\Phi(\mathbb{1}_A) = \mathbb{1}_B, \Phi(ab) = \Phi(a)\Phi(b), \Phi(a^*) = \Phi(a)^*.$$

We now start building such Φ on the algebra of continuous functions on $\sigma(A)$, $C(\sigma(A))$.

5.11.1 The map Φ_A on polynomials

Lemma 19. *Let H be a complex Hilbert space, $A \in L(H)$. For every $p(z) = \sum_{k=0}^n a_k z^k$,*

$$p(A) = \sum_{k=0}^n a_k A^k \in L(H)$$

has the properties

$$(p+q)(A) = p(A) + q(A), pq(A) = p(A)q(A), p(\sigma(A)) = \sigma(p(A)), \|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|.$$

The next technical result is the (general) Stone-Weierstrass theorem.

Theorem 32. *(Stone-Weierstrass) Let M be a Hausdorf compact space and $A \subset C(M)$ be a subalgebra, with the following properties*

1. $1 \in A$
2. *A separates points on M, i.e. for every $x, y \in M, x \neq y$, there exists $f \in A$, so that $f(x) \neq f(y)$.*
3. *A is closed under conjugation, i.e. if $f \in A$, then $\bar{f} \in A$.*

Then, A is dense in $C(M)$.

5.11.2 The construction of the map $\Phi_A : C(\Sigma) \to L(H)$

Theorem 33. *Let H be a complex Hilbert space, $A \in L(H)$ be self-adjoint operator, i.e. $A = A^*$. Let $\Sigma = \sigma(A)$. Then, there exists a bounded, complex linear operator*

$$\Phi_A : C(\Sigma) \to L(H), f \to f(A),$$

so that

- **Product** $\Phi_A(\mathbb{1}) = Id$, $fg(A) = f(A)g(A)$.
- **Conjugation** $\bar{f}(A) = f(A)^*$
- **Normalization** *If* $f(\lambda) = \lambda$, *then* $f(A) = A$
- **Isometry** $\|f(A)\| = \sup_{\lambda\in\Sigma}|f(\lambda)|$
- **Commutative** *If* $AB = BA$, *then* $f(A)B = Bf(A)$ *for all* $f \in C(\Sigma)$.
- **Image**
$$\mathscr{A} := \{f(A) : f \in C(\sigma) = \cap\{\mathscr{B} : \mathscr{B}\ C^*\ algebra \subset L(H), A \in \mathscr{B}\}$$
- **Eigenvector** *If* $Ax = \lambda x$, *for some* $\lambda \in \mathbb{C}$ *and* $x \in H$, *then* $f(A)x = f(\lambda)x$.
- **Spectrum** $f(A)$ *is normal for all* $f \in C(\Sigma)$, $\sigma(f(A)) = f(\sigma(A))$.
- **Composition** *For* $f \in C(\Sigma, \mathbf{R})$, $g \in C(f(\Sigma))$, $g \circ f(A) = g(f(A))$.

5.11.3 Square roots

Definition 28. *We say that* $A \in L(H) : A = A^*$ *is positive semi-definite, if* $\langle Ax, x\rangle \geq 0$. *We denote it* $A \geq 0$.

We have the following theorem.

Theorem 34. *Let* H *be a complex Hilbert space and* $A = A^*$. *Let* $f \in C(\sigma(A))$. *Then,*

1. $f(A) = f(A)^*$ *if and only if* $f(\sigma(A)) \subset \mathbf{R}$.
2. *Assume* $f(\sigma(A)) \subset \mathbf{R}$. *Then,* $f(A) \geq 0$ *if and only if* $f \geq 0$.
3. $A \geq 0$ *if and only if there exists* $B = B^*$, *so that* $A = B^2$.

Proof. The proof of 1) is easy, see p. 245. For 2), we have

$$\inf_{\|x\|=1}\langle f(A)x, x\rangle = \inf\sigma(f(A))) = \inf f(\sigma((A))) = \inf_{\lambda\in\sigma(A)} f(\lambda)$$

So, $f(\lambda) \geq 0$ if and only if $f(A) \geq 0$.

For 3), assume that $A \geq 0$. Then, $\sigma(A) \subset [0, \infty)$. Then, consider the function $f(\lambda) = \sqrt{\lambda}$, which is well-defined. Then, $B := f(A) \in L(H)$. Also, since $f^2(\lambda) = \lambda$,

$$B^2 = f^2(A) = A.$$

The reverse direction is easy, since if $A = B^2$,

$$\langle Ax, x \rangle = \langle B^2 x, x \rangle = \langle Bx, Bx \rangle = \|Bx\|^2 \geq 0.$$

□

5.12 Spectral Measures

Assume H is a nonzero complex Hilbert space and $A \in L^c(H)$ is a normal operator, then the spectrum $\Sigma = \sigma(A) \subset \mathbb{C}$. This section is to assign to A a Borel measure on Σ with values in the space of orthogonal projections on H, called the **spectral measure of** A.

When A is compact, the spectral measure of A assigns to each Borel set $\Omega \subset \Sigma$ the projection $P_\Omega := \sum_{\lambda \in \sigma(A) \cap \Omega} P_\lambda$.

Definition 29. *(**Projection Valued Measure**) Let H be a complex Hilbert space, let $\Sigma \subset \mathbb{C}$ be nonempty and closed, and denote by $\mathscr{B} \subset 2^\Sigma$ the Borel σ-algebra. A* **projection valued Borel measure** *on Σ is a map $\Omega \to P_\Omega$, where P_Ω is a bounded complex linear operator on H and satisfies*

- *(**Projection**) For $\forall\ \Omega \subset \Sigma$, $P_\Omega^2 = P_\Omega = P_\Omega^*$.*
- *(**Normalization**) $P_\emptyset = 0$ and $P_\Sigma = \mathbb{1}$.*
- *(**Intersection**) If $\Omega_1, \Omega_2 \subset \Sigma$, then $P_{\Omega_1 \cap \Omega_2} = P_{\Omega_1} P_{\Omega_2} = P_{\Omega_2} P_{\Omega_1}$.*
- *(σ-**Additive**) If $(\Omega_i)_{i \in \mathbb{N}} \subset \Sigma$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\Omega = \cup_{i=1}^\infty \Omega_i$, then $P_\Omega x = \lim_{n \to \infty} \sum_{i=1}^n P_{\Omega_i} x$ for all $x \in H$.*

For every nonempty compact Hausdorff space Σ, define

$$B(\Sigma) = \{f : \Sigma \to \mathbb{C} | f \text{ is bounded and Borel measurable}\}. \tag{14}$$

It is a C^* algebra with the norm $\|f\| = \sup_{\lambda \in \Sigma} |f(\lambda)|$ for $f \in B(\Sigma)$.

Theorem 35. *Let $H, \Sigma, \mathscr{B}$ be as in Definition 29. $B(\Sigma)$ is defined in (14). For $x, y \in H$, define the signed Borel measure $\mu_{x,y} : \mathscr{B} \to \mathbb{R}$ by*

$$\mu_{x,y}(\Omega) = \Re\langle x, P_\Omega y\rangle. \tag{15}$$

Then for each $f \in B(\Sigma)$, $\exists!$ operator $\Psi(f) \in L^c(H)$ such that for all $x, y \in H$,

$$\langle x, \Psi(f) y\rangle = \int_\Sigma \Re f \, d\mu_{x,y} + \int_\Sigma Im f \, d\mu_{x,iy}. \tag{16}$$

The map $\Psi : B(\Sigma) \to L^c(H)$ is a C^* algebra homomorphism and a bounded linear operator and $\sigma(\Psi(f)) \subset \overline{f(\Sigma)}$.

Proof. Denote $\mathscr{M}(\Sigma)$ by the Banach space of signed Borel measure $\mu : \mathscr{B} \to \mathbb{R}$ with $\|\mu\| = \sup_{\Omega\in\mathscr{B}}(\mu(\Omega) - \mu(\Sigma\backslash\Omega))$. The proof has five steps.

Step 1. The map $H \times H \to \mathscr{M}(\Sigma) : (x, y) \mapsto \mu_{x,y}$ is real bilinear and symmetric and satisfies $\|\mu_{x,y}\| \leq \|x\| \|y\|$.

The vectors $P_\Omega y$ and $P_{\Sigma\backslash\Omega} y$ are orthogonal to each other. Hence

$$\|P_\Omega y - P_{\Sigma\backslash\Omega} y\|^2 = \|P_\Omega y\|^2 + \|P_{\Sigma\backslash\Omega} y\|^2 = \|P_\Omega y + P_{\Sigma\backslash\Omega} y\|^2 = \|P_\Sigma y\|^2 = \|y\|^2 \tag{17}$$

Hence by (15), $\mu_{x,y}(\Omega) - \mu_{x,y}(\Sigma\backslash\Omega) = \Re\langle x, (P_\Omega - P_{\Sigma\backslash\Omega})y\rangle \leq \|x\| \|y\|$.

Step 2. Let $B \in L^c(H)$ such that $P_\Omega B = BP_\Omega$ for all $\Omega \in \mathscr{B}$. Then $\mu_{x,By} = \mu_{B^*x,y}$ for all $x, y \in H$.

By (15), we have

$$\mu_{x,By}(\Omega) = \Re\langle x, P_\Omega By\rangle = \Re\langle x, BP_\Omega y\rangle = \Re\langle B^* x, P_\Omega y\rangle = \mu_{B^*x,y}(\Omega).$$

Step 3. For every $f \in B(\Sigma)$, $\exists!$ a bounded complex linear operator $\Psi(f)$ that satisfies (16) and $\Psi(\overline{f}) = \Psi(f)^*$ and Ψ is a bounded complex linear operator.

Define the real bilinear form $B_f : H \times H \to \mathbb{R}$ by $B_f(x, y) = \int_\Sigma f d\mu_{x,y}$. Then $|B_f(x, y)| \leq \|f\| \|\mu_{x,y}\| \leq \|f\| \|x\| \|y\|$ by Step 1. Hence $\exists!$ bounded real linear operator $\Psi(f) \in L(H)$ such that $\Re\langle x, \Psi(f)y\rangle = B_f(x, y)$. Moreover, $B_f(x, iy) = -B_f(ix, y)$ by Step 2 with $B = i\mathbb{1}$, hence

$$\Re\langle x, \Psi(f)iy\rangle = B_f(x, iy) = -B_f(ix, y) = -\Re\langle ix, \Psi(f)y\rangle = \Re\langle x, i\Psi(f)y\rangle.$$

Thus $\Psi(f)$ is complex linear.

Define $\Psi(f) = \Psi(\Re f) + i\Psi(\mathrm{Im} f) \in L^c(H)$, then (16) holds. Since the operators $\Psi(\Re f)$ and $\Psi(\mathrm{Im} f)$ are self-adjoint, $\Psi(\overline{f}) = \Psi(f)^*$.

Step 4. Let Ψ be as in Step 3. Then $\Psi(fg) = \Psi(f)\Psi(g)$ for all $f, g \in B(\Sigma)$.

It suffices to show the case for real valued functions $f, g \in B(\Sigma, \mathbb{R})$. Assume $g = \chi_\Omega$, then for all $\Omega' \in \mathscr{B}$,

$$\mu_{P_\Omega x, y}(\Omega') = \Re\langle P_\Omega x, P_{\Omega'} y\rangle = \Re\langle x, P_\Omega P_{\Omega'} y\rangle = \Re\langle x, P_{\Omega\cap\Omega'} y\rangle$$
$$= \mu_{x,y}(\Omega\cap\Omega') = \int_{\Omega'} \chi_\Omega d\mu_{x,y}$$

For all $g \in B(\Sigma, \mathbb{R})$,

$$\int_\Omega g d\mu_{x,y} = \int_\Sigma g\chi_\Omega \, d\mu_{x,y} = \int_\Sigma g\mu_{P_\Omega x, y} = \Re\langle P_\Omega x, \Psi(g) y\rangle$$
$$= \Re\langle x, P_\Omega \Psi(g) y\rangle = \mu_{x, \Psi(g) y}(\Omega)$$

Further, for all $f, g \in B(\Sigma, \mathbb{R})$,

$$\Re\langle x, \Psi(fg) y\rangle = \int_\Sigma fg \, d\mu_{x,y} = \int_\Sigma f \, d\mu_{x,\Psi(g)y} = \Re\langle x, \Psi(f)\Psi(g) y\rangle.$$

Step 5. $\sigma(\Psi(f)) \subset \overline{f(\Sigma)}$.

We prove if $\lambda \in \mathbb{C}\backslash\overline{f(\Sigma)}$, then $\lambda \in \rho(\Psi(f))$. FOr this consider the function $g(\mu) = (\lambda - f(\mu))^{-1} \in B(\Sigma)$. Since $g(\mu)(\lambda - f(\mu)) = 1$,

$$g(A)(\lambda - f(A)) = Id,$$

whence $\lambda \in \rho(A)$. □

Examples: Let $\Sigma \subset \mathbb{C}$ be nonempty and compact, let $H = L^2(\Sigma)$. Define $P_\Omega \psi = \chi_\Omega \psi$ for $\psi \in H$, where χ denotes the characteristic function of Ω. Define Ψ in Theorem 35 by $\Psi(f)\psi = f\psi$. In the case $\Sigma = [0,1]$ and $f(\lambda) = \lambda$ when $0 \leq \lambda < 1$ and $f(1) = 2$. We obtain $f(\Sigma) = [0,1) \cup \{2\}$ and $\sigma(\Psi(f)) = [0,1]$. Thus $f(\Sigma)$ is not closed and $f(\Sigma) \not\subset \sigma(\Psi(f)) \subsetneq \overline{f(\Sigma)}$.

5.13 Measurable Functional Calculus

The next theorem extends the continuous functional calculus for normal operators to bounded measurable functions.

Theorem 36. *Let H be a complex Hilbert space, $A \in L(H) : A = A^*$ be a self-adjoint operator. Let $\Sigma = \sigma(A)$. Then, there exists a complex linear operator*

$$\Phi_A : B(\Sigma) \to L(H), f \to f(A),$$

so that

- **Product** $1(A) = \mathbb{1}$, $(fg)(A) = f(A)g(A)$.
- **Conjugation** $\bar{f}(A) = f(A)^*$
- **Positive** *If* $f \geq 0$, *then* $f(A) = f(A^*) \geq 0$.
- **Normalization** *If* $f(\lambda) = \lambda$, *then* $f(A) = A$.
- **Contraction** $\|f(A)\| \leq \sup_{\lambda \in \Sigma} |f(\lambda)| = \|f\|$.
- **Convergence** *Let* $f_i \in B(\Sigma)$ *with* $\sup_{i \in \mathbb{N}} \|f_i\| < \infty$, *and* $\lim_{i \to \infty} f_i(\lambda) = f(\lambda)$, $f \in B(\Sigma)$. *Then* $\lim_{i \to \infty} f_i(A)x = f(A)x$.
- **Commutative** *If* $AB = BA$ *and* $A^*B = BA^*$, *then* $f(A)B = Bf(A)$ *for all* $f \in B(\Sigma)$.
- **Eigenvector** *If* $Ax = \lambda x$, *for some* $\lambda \in \mathbb{C}$ *and* $x \in H$, *then* $f(A)x = f(\lambda)x$.
- **Spectrum** $f(A)$ *is normal for all* $f \in B(\Sigma)$, *and* $\sigma(f(A)) \subset \overline{f(\sigma(A))}$. *For all* $f \in C(\Sigma)$, $\sigma(f(A)) = f(\sigma(A))$.
- **Composition** *For* $f \in C(\Sigma, \mathbf{R})$, $g \in B(f(\Sigma))$, $g \circ f(A) = g(f(A))$.

Proof. By Theorem 33, for each $x \in H$, there exists a unique Borel measure $\mu_x : \mathscr{B} \to [0, \infty)$ such that $\int_\Sigma f \, d\mu_x = \langle x, f(A)x \rangle$. We define the signed measure $\mu_{x,y} : \mathscr{B} \to \mathbb{R}$ by

$$\mu_{x,y} = \frac{1}{4}(\mu_{x+y} - \mu_{x-y}) \tag{18}$$

For $f \in C(\Sigma)$ and real-valued, is as follows.

$$\Re\langle x, f(A)y \rangle := \frac{\langle x+y, f(A)(x+y) \rangle - \langle x-y, f(A)(x-y) \rangle}{4} = \int_\Sigma f \, d\mu_{x,y}$$

On the other hand, the bilinear form $B_f(x,y) = \int_\Sigma f\, d\mu_{x,y}$ is symmetric and bounded. So, there is a bounded symmetric operator, call it $\Psi_A(f)$, so that

$$\int_\Sigma f\, d\mu_{x,y} = B_f(x,y) = \Re\langle x, \Psi_A(f) y\rangle.$$

Clearly $\Psi_A(f) = f(A)$ for $f \in C(\Sigma)$ and real-valued. Most properties are trivial to check directly, but oespecially so, if one has the convergence property. All one has to do is

- acknowledge that any function $f \in B(\Sigma)$ is approximated point-wise by the smooth functions, $f_n := f * \chi_n = n \int f(\cdot - y)\chi(ny)dy$, where χ is a smooth compactly supported function, $\int \chi(x)dx = 1$. Note $\sup_n \|f_n\|_{C(\Sigma)} \leq \|f\|_{L^\infty}$.
- take into account the corresponding properties for the continuous functional calculus.
- All properties hold for $f_n \in C(\Sigma)$. Take limits, so they hold for f as well.

We are now checking the convergence property.

Let $f_n : \sup_n \|f_n\|_{L^\infty} < \infty$ and $f_n \to f$ point-wise. Then, for each $x, y \in H$, we have by the Lebesgue dominated convergence theorem (Σ is compact, so $\sup_n \|f_n\|_{L^\infty} < \infty$ is enough domination)

$$\langle y, \Psi_A(f)x\rangle = \int_\Sigma f d\mu_{y,x} = \lim_n \int_\Sigma f_n d\mu_{y,x} = \lim_n \langle y, \Psi_A(f_n)x\rangle.$$

This means that $\Psi_A(f_n)x$ converges <u>weakly</u> to $\Psi_A(f)x$.

But now, again by the Lebesgue dominated convergence theorem

$$\begin{aligned} \|\Psi_A(f)x\|^2 &= \langle \Psi_A(f)x, \Psi_A(f)x\rangle = \langle x, \Psi_A(f)^*\Psi_A(f)x\rangle = \langle x, \Psi_A(\overline{f})\Psi_A(f)x\rangle \\ &= \langle x, \Psi_A(|f|^2)x\rangle = \int_\Sigma |f|^2 d\mu_x = \lim_n \int_\Sigma |f_n|^2 d\mu_x = \lim_n \|\Psi_A(f_n)x\|^2 \end{aligned}$$

It follows that $\Psi_A(f_n)x$ converges weakly to $\Psi_A(f)x$ and in addition $\lim_n \|\Psi_A(f_n)x\| = \|\Psi_A(f)x\|$. In Hilbert spaces, this implies $\lim_n \|\Psi_A(f_n)x - \Psi_A(f)x\|_H = 0$, so convergence is checked. $\square$

5.14 Cyclic Vectors

The spectral measure can be used to identify a self-adjoint operator on a Hilbert space with a multiplication operator.

Definition 30. *(**Cyclic Vector**) Let H be a complex Hilbert space and let $A = A^* \in L^c(H)$. A vector $x \in H$ is called cyclic for A if $H = \overline{span\{A^n x | n = 0, 1, 2, ...\}}$. If such x exists, then clearly H is separable.*

The next result is the **multiplicative form of the spectral theorem for Hilbert spaces with cyclic vectors**.

Theorem 37. *Let H be a complex Hilbert space, let $A = A^* \in L^c(H)$. Let $\Sigma = \sigma(A) \subset \mathbb{R}$, and let $B \subset 2^\Sigma$ be the Borel σ-algebra and $x \in H$ be a cyclic vector for A. Then the following holds.*

(1) There is a unique Hilbert space isometry $U : H \to L^2(\Sigma, \mu_x)$ such that $U^{-1}\psi = \psi(A)x$ for all $\psi \in C(\Sigma)$.

(2) Let f be a bounded Borel measurable function. Then $Uf(A)U^{-1}\psi = f\psi$ for all $\psi \in L^2(\Sigma, \mu_x)$.

(3) U in part (1) satisfies $(UAU^{-1}\psi)(\lambda) = \lambda\psi(\lambda)$.

(4) If $\Omega \subset \Sigma$ is nonempty and (relatively) open, then $\mu_x(\Omega) > 0$.

Basically, the theorem says that A is unitarily equivalent to a multiplication operator (by x) on an L^2 based space.

Examples:

- Let μ be a Borel measure on Σ such that every nonempty relatively open subset of Σ has positive measure. Define $A : L^2(\Sigma, \mu) \to L^2(\Sigma, \mu)$ by $(A\psi)(\lambda) = \lambda\psi(\lambda)$, for $\lambda \in \Sigma$. Then A is self-adjoint and $\sigma(A) = \Sigma$.

- Let H be a complex Hilbert space, let $A = A^* \in L^c(H)$. Then A admits a cyclic vector if and only if A is injective and $E_\lambda = \ker(\lambda\mathbb{1} - A)$ has dimension one for $\lambda \in P\sigma(A)$.

- Let $A = A^* \in \mathbb{C}^{n\times n}$ be a Hermitian matrix and $e_1,...e_n$ be an orthonormal basis of eigenvectors, so $Ae_i = \lambda_i e_i$. Thus $\Sigma = \sigma(A) = \{\lambda_1,...\lambda_n\}$. Assume $\lambda_i \neq \lambda_j$ for $i \neq j$. Then
  - $f(A)x = \sum_{i=1}^n f(\lambda_i)\langle e_i, x\rangle e_i$ for $f : \Sigma \to \mathbb{C}$.
  - $x = \sum_{i=1}^n e_i$ is a cyclic vector and that $\mu_x = \sum_{i=1}^n \delta_{\lambda_i}$, so $\int_\Sigma f\, d\mu_x = \sum_{i=1}^n f(\lambda_i)$.
  - Let U be the isometry in Theorem 37, then $(Ux)(\lambda_i) = \langle e_i, x\rangle$ and $U^{-1}\psi = \sum_{i=1}^n \psi(\lambda_i)e_i$ for $\psi \in L^2(\Sigma,\mu_x)$.
- Let H be an infinite-dimensional separable complex Hilbert space and let $A = A^* \in L^c(H)$. Assume that $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of A, so $Ae_i = \lambda_i e_i$, $\lambda_i \in \mathbb{R}$. Thus $\sup_{i\in\mathbb{N}} |\lambda_i| < \infty$ and $\Sigma = \overline{\{\lambda_i | i \in \mathbb{N}\}}$. Assume $\lambda_i \neq \lambda_j$ for $i \neq j$. Then
  - $f(A)x = \sum_{i=1}^\infty f(\lambda_i)\langle e_i, x\rangle e_i$, for any bounded $f : \Sigma \to \mathbb{C}$.
  - If $\sum_{i=1}^\infty \epsilon_i^2 < \infty$, then $x = \sum_{i=1}^\infty \epsilon_i e_i$ is cyclic for A and $\mu_x = \sum_{i=1}^\infty \epsilon_i^2 \delta_{\lambda_i}$.
  - Then the map $\psi \to (\psi(\lambda_i))_{i\in\mathbb{N}}$ defines an isomorphism
  $$L^2(\Sigma,\mu_x) \cong \{\eta = (\eta_i)_{i=1}^\infty | \sum_{i=1}^\infty \epsilon_i^2 |\eta_i|^2 < \infty\}.$$
  Then U is given by $(U^{-1}\psi) = \sum_{i=1}^\infty \epsilon_i \psi(\lambda_i)e_i$. Then the operator $\Lambda := UAU^{-1}$ is given by $\eta \mapsto (\lambda_i \eta_i)_{i\in\mathbb{N}}$.
- Consider the Hilbert space
  $$H = l^2(\mathbb{Z},\mathbb{C}) = \{x = (x_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} | \sum_{n=-\infty}^\infty |x_n|^2 < \infty\},$$
  and define $A : H \to H$ by $Ax = (x_{n-1} + x_{n+1})_{n\in\mathbb{Z}}$ for $x = (x_n)_{n\in\mathbb{Z}} \in H$. Thus $A = L + L^*$, where $Lx = (x_{n+1})_{n\in\mathbb{Z}}$. $e_i = (\delta_{in})_{n\in\mathbb{Z}}$, $i \in \mathbb{Z}$ form an orthonormal basis of H.
  - Define $\Phi : H \to L^2([0,1])$ by $(\Phi x)(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n t} x_n$. Then Φ is an isometric isomorphism and
  $$(\Phi A \Phi^{-1} f)(t) = 2\cos(2\pi t) f(t)$$
  - $P\sigma(A) = \emptyset$ and $\Sigma = \sigma(A) = [-2,2]$.

6 Unbounded Operators

6.1 Unbounded Operators on Banach Spaces

Definition 31. *(Unbounded Operator)*

Let X and Y are Banach spaces (real or complex). A pair (A, dom(A)) is called an unbounded linear operator. Here, $A : dom(A) \to Y$ is a linear map and $dom(A) \subset X$ is a linear space.

A is **densely defined** *if its domain is a dense subspace of X. A is* **closed** *if its graph is a closed linear subpace of $X \times Y$, where $graph(A) = \{(x, Ax) | X \in dom(A)\}$.*

Equivalently, A is closed, if whenever $x_n \in D(A)$ and $(x_n, Ax_n) \to (x, y)$, then $x \in D(A)$ and $y = Ax$.

Remark: The case $\mathrm{dom}(A) = X$ is included in the Definition (31). In this case, the Closed Graph Theorem implies that A has a closed graph if and only if A is bounded. We focus on the case that the domain dom(A) is proper subspace in X.

Definition 32. *We say that A is* **closeable***, if there exist an extension $\tilde{A}$ that is a closed operator. That is, there is $D(\tilde{A}) \supset D(A)$ and $\tilde{A}|_{D(A)} = A$.*

Proposition 15. *$A : D(A) \subset X \to Y$ is closeable if and only if for all $x_n \in D(A), y \in X$, so that $x_n \to 0, Ax_n \to y$, we have $y = 0$.*

Examples:

- Let $X = C([0,1])$. Then $Af := f', \quad \mathrm{dom}(A) = C^1([0,1])$ defines an unbounded operator with a dense domain and a closed graph.
- Let H be a separable Hilbert space. Define the operator

$$A_\lambda : \mathrm{dom}(A_\lambda) \to H, \quad A_\lambda x := \sum_{i=1}^{\infty} \lambda_i \langle e_i, x \rangle e_i,$$

  where $\mathrm{dom}(A_\lambda) = \{x \in H : \sum_{i=1}^{\infty} |\lambda_i \langle e_i, x \rangle|^2 < \infty\}$ and $(e_i)_{i \in \mathbb{N}}$ is an orthonormal basis. It is an unbounded operator with a dense domain and a close graph.

- Let $X = L^p(\mathbb{R}, \mathbb{C})$. Define the operator

$$Af = \frac{df}{ds}, \quad \mathrm{dom}(A) = W^{1,p}(\mathbb{R}, \mathbb{C}) = \{f \in L^p : f' \in L^p\}.$$

  The operator has a closed graph and it has a dense domain for $1 \le p < \infty$.

- Let $1 < p < \infty$. The **Laplace operator** $\Delta : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ has a closed graph.

- Let $1 \le p < \infty$ and (M, A, μ) be a measure space. **The multiplication Operator**

$$A_f : \mathrm{dom} A_f \to L^p(\mu) \quad A_f \psi := f\psi, \quad \mathrm{dom}(A_f) = \{\psi \in L^p(\mu), f\psi \in L^p(\mu)\}.$$

  has a dense domain and a closed graph.

6.2 The Spectrum of an Unbounded Operator

We restrict the discussion to operators with closed graphs in this section.

Definition 33. *Let $A : dom(A) \to X$ be an unbounded linear operator with a closed graph. The* **spectrum** *of A is defined as $\sigma(A) = P\sigma(A) \cup R\sigma(A) \cup C\sigma(A)$, here $P\sigma(A), R\sigma(A)$ and $C\sigma(A)$ are defined in Chapter 5.*

The next lemma is about the resolvent operator. The resolvent identity of Lemma (16) also holds for unbounded operators.

Lemma 20. *Let X be a complex Banach space and let $A : dom(A) \to X$ be an unbounded linear operator.*

- *Let $\mu \in \rho(A)$, $\lambda \in \mathbb{C}$ such that $|\lambda - \mu| \|(\mu - A)^{-1}\| < 1$. Then $\lambda \in \rho(A)$ and*

$$(\lambda - A)^{-1} = \sum_{k=0}^{\infty} (\mu - \lambda)^k (\mu - A)^{-k-1}.$$

- *Let $\lambda, \mu \in \rho(A)$, then $R_\lambda(A)$, $R_\mu(A)$ commute and*

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda) R_\lambda(A) R_\mu(A).$$

Definition 34. *Let X be a complex Banach space and let A be an unbounded operator with a closed graph. If $\rho(A) \neq \emptyset$ and $R_\lambda(A) = (\lambda - A)^{-1} \in L(X)$ is compact for all $\lambda \in \rho(A)$, then A is said to have a compact resolvent.*

Theorem 38. *Let X be a complex Banach space and let A be an unbounded operator with compact resolvent. Then $\sigma(A) = P\sigma(A)$ is a discrete subset of $\mathbb{C} = \{\lambda_n\}$, without finite point of accumulation and the property $\lim_n |\lambda_n| = \infty$. Finally,* **the generalized eigenspace corresponding to an eigenvalue** *λ, $E_\lambda := \cup_{k=1}^{\infty} \ker(\lambda - A)^k$ is finite dimensional.*

6.3 Spectral Projections

Definition 35. *Let A be an unbounded complex operator with a closed graph on a complex Banach space X and let $\Sigma \subset \sigma(A)$ be compact. If $\sigma(A)\backslash\Sigma$ is a closed subset of $\mathbb{C}$, then Σ is called to be isolated, if there is an open set $U \subset \mathbb{C}$, a neighborhood of Σ, so that $\sigma(A) \cap U = \Sigma$. Let γ be a curve in U, so that Σ belongs to its interior. In addition, we require that*

$$ind_\gamma(\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\lambda} = \begin{cases} 1 & \lambda \in \Sigma \\ 0 & \lambda \in \mathbf{C} \setminus U \end{cases}$$

We define the operator $\phi(f) \in L(X)$,

$$\phi_A(f) = \frac{1}{2\pi i}\int_\gamma f(z)(z-A)^{-1}\, dz.$$

Heuristically, $\phi_A(f) = f(A)$.

Theorem 39. *Let X, A, Σ, U be as in Definition 35. Then, for every $f, g \in H(U)$,*

(1) $\phi(f+g) = \phi(f) + \phi(g)$ *and* $\phi(fg) = \phi(f)\phi(g)$.

(2) $\sigma(\phi(f)) = f(\Sigma)$.

(3) $g(\phi(f)) = \phi(g \circ f)$.

(4) The operator

$$\mathbb{P}_\Sigma := \phi(\mathbb{1}_\Sigma) = \frac{1}{2\pi i}\int_\gamma (z-A)^{-1}\, dz$$

is a projection operator. $X_\Sigma = Im\mathbb{P}_\Sigma$ *is an invariant subspace for* A*. In addition* $A_\Sigma := A\mathbb{P}_\Sigma : X_\Sigma \to X_\Sigma$ *is a bounded operator, with* $\sigma(A_\Sigma) = \Sigma$*. Finally, the holomorphic functional calculus for the bounded operator* A_Σ *is as follows* $f(A_\Sigma) = \phi_A(f\mathbb{1}_\Sigma)$*.*

Lemma 21. *If* $A : D(A) \subset X \to Y$ *is closed operator, then* $Ker(A) = \{x \in D(A) : Ax = 0\}$ *is a closed subspace of* X*.*

6.4 Dual operators

We discuss dual operators in this section.

Definition 36. *Let* X *and* Y *be Banach spaces(real or complex), and let* A *be an unbounded operator with a dense domain* $D(A)$*. The domain of the dual operator is defined as follows*

$$D(A^*) = \{y^* \in Y^* : \forall x \in D(A) : |\langle Ax, y^* \rangle| \leq C\|x\|_X\}.$$

In such a case, by the density of $D(A)$*, one can find an unique functional* $x^* \in X^*$*, so that* $\langle x^*, x\rangle = \langle y^*, Ax\rangle$*. We set* $A^* y^* := x^*$*. In other words, the dual operator is defined*

$$\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle\,.$$

Theorem 40. *(Basic properties of the adjoint operator) Let* X, Y, A *be as in Definition 36. Then*

(1) A^* *is closed.*

(2) Let $x \in X$ *and* $y \in Y$*. Then*

$$(x, y) \in \overline{graph(A)} \Longleftrightarrow \langle y^*, y\rangle = \langle A^* y^*, x\rangle\,, \quad \forall y^* \in D(A^*).$$

(3) A *is closeable* $\Longleftrightarrow D(A^*)$ *is weak** *dense in* Y^**.*

(4) $Im(A)^\perp = \ker(A^*)$*. If* A *has a closed graph, then* ${}^\perp Im(A^*) = \ker(A)$*.*

(5) A *has a dense image* $\Longleftrightarrow A^*$ *is injective.*

(6) If A has a closed graph. Then

$$A \text{ is injective} \Longleftrightarrow A^* \text{ has a} \textit{weak}^* \text{ dense image.}$$

Theorem 41. *(Closed Image Theorem for unbounded operators) Let X, Y, A be as in Definition 36, and A is closed, then the following are equivalent.*

(1) $Im(A) = {}^{\perp}\ker(A^*)$.

(2) $Im(A)$ *is a closed subspace of* Y.

(3) $\exists c > 0$,

$$\inf_{A\xi=0} \|x+\xi\| \le c\|Ax\|, \ \textit{for all } x \in D(A).$$

(4) $Im(A^*) = \ker(A)^{\perp}$.

(5) $Im(A^*)$ *is a weak** *closed subpace of* X^*.

(6) $Im(A^*)$ *is a closed subspace of* X^*.

(7) $\exists c > 0$,

$$\inf_{A^*\eta^*=0} \|y^*+\eta^*\| \le c\|A^*y^*\|, \ \textit{for all } y^* \in D(A^*).$$

Remark: We have identical theorem for bounded operators, Theorem 19. Clearly, Theorem 19 is a corollary of Theorem 41, as one can apply Theorem 41 to a bounded operator A. On the other hand, Theorem 19 plays a big part in the proof of Theorem 41, especially in establishing that having $Im(A^*)$ closed implies that $Im(A)$ is closed.

Corollary 9. *Let X, Y, A be as in Definition* (36)*, and A is closed. Then A is bijective if and only if A^* is bijective. Then $A^{-1} : Y \to X$ is bounded and* $(A^*)^{-1} = (A^{-1})^*$.

A Finite dimensional subspaces of a Banach space

We show that every finite dimensional subspace of a (real) Banach space is isomorphic to $\mathbf{R}^n$.

Proposition 16. *Let X be a real Banach space and $x_1, \dots, x_n$ be a finite family of linearly independent vectors in X. Then, $X_n = \overline{span[x_1, \dots, x_n]}$ is a Banach space isomorphic to $\mathbf{R}^n$. In particular,*

$$c\|\sum_{j=1}^{n} \lambda_j x_j\|_X \leq \max_{1\leq j\leq n} |\lambda_j| \leq C\|\sum_{j=1}^{n} \lambda_j x_j\|_X \tag{19}$$

B Dual families

Th next Proposition provides the existence of a finite dual family of vectors.

Proposition 17. *(see Corollary 2.3.4) Let X be a real Banach space and $x_1, \dots, x_n$ be a finite family of linearly independent vectors in X. Then, there exists a family of vectors $x_1^*, \dots, x_n^* \in X^*$, so that*

$$\langle x_j^*, x_i\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Remark: This is a generalization of a corollary of Hahn-Banach, which we use frequently. Namely, for every $x \in X$, there is $x^* \in X^* : \|x^*\| = 1$, so that $\langle x^*, x\rangle = \|x\|$, see Corollary 6. Here, we have a family of vectors, whose norms are unknown, but keep in mind that it is usually the case that $\max_{1\leq j\leq n} \|x_j^*\|_{X^*}$ is growing with n.

C Finite dimensional subspaces/finite co-dimension subspace of a Banach space are complemented

The next result is almost an immediate consequence of Proposition 17, see also the construction of K in the proof of Lemma 11.

Proposition 18. *Let X be a Banach space and X_0 be a finite dimensional subspace of X. Then, X_0 is complemented. That is, there exists a closed linear subspace $Y \subset X$, so that*

$$X = X_0 \oplus Y.$$

Equivalently, there is a projection operator $P : X \to X_0$, i.e. an $P^2 = P$. Note that $Q = I - P$ also satisfies $Q^2 = Q$ and then, $Y = Im(Q)$.

Let Y be a finite co-dimension subspace of X, i.e. $dim(X/Y) < \infty$. Then, Y is complemented.

D Small perturbations of invertible operators are still invertible

Lemma 22. *Let $A : X \to Y$ be bounded and invertible linear operator. Then, there exists $\epsilon = \epsilon(A)$ so that for each $B : X \to Y$, $\|B\| < \epsilon$, we have that $A + B$ is also invertible.*

Remark: In fact, one may take $\epsilon = \frac{1}{\|A^{-1}\|}$.

References

[1] T. Bühler and D. Salamon, Functional Analysis (Graduate Studies in Mathematics), American Mathematical Society, **191** Providence, RI, 2018.