MATH 961: PROJECT III
DUE: NOV. 5TH, 2020

(1) Prove that
  - If C is symmetric on a complex Hilbert space H, then $Ker(C+i) = \{0\}$. **Hint:** $\langle Cz + iz, z\rangle = 0$.
  - If $A \subset C$ (i.e. C is an extension of A) are symmetric operators, with $Ran(A+i) = Ran(C+i)$, show that $C = A$.
  - Show that if A is symmetric, with $Ran(A+i) = H$, but $Ran(A-i) \neq H$, then A does not have any self-adjoint extensions.

(2) Consider $T = i\partial_x$ on $L^2(0,\infty)$, with $D(T) = C_0^\infty([0,\infty))$, the functions with compact support away from the origin. Show that T is symmetric, compute T^*.

(3) Show that the operators

$$S(t)f(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} f(x-y)dy$$

representing the solution of the heat equation $u_t = u_{xx}, x \in \mathbb{R}^1$, $u(t,x)$ with initial data $f(x)$ for a C_0 semi-group on $L^p(\mathbb{R}^1), 1 \leq p < \infty$.

More specifically, check the property $\lim_{t\to 0+}\|S(t)f - f\|_{L^p} = 0$ for each fixed $f \in L^p$. Why this fails at $p = \infty$? Show that this is a semigroup of contractions in the same spaces.

(4) Let H be a Hilbert space, with an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Let $\{a_n\} \subset \mathbb{C}$. Let

$$Ax := \sum_n a_n\langle x, e_n\rangle e_n, D(A) = \{x \in H : \sum_n |a_n|^2|\langle x, e_n\rangle|^2 < \infty.$$

Show that A generates a C_0 semigroup if and only if $\sup_n \Re a_n < \infty$. How does the semi-group $T(t) = e^{tA}$ look like? Compute its growth bound $\omega = \lim_{t\to\infty}\frac{\ln\|T(t)\|}{t}$.

(5) For $1 < p < \infty$, define the space

$$W_p^1(0,1) = \{f \in C(0,1)\cap L^p(0,1) : \exists g \in L^p(0,1) : f(t) - f(s) = \int_s^t g(\tau)d\tau\}$$

with norm $\|f\|_{W_p^1} := \|g\|_{L^p(0,1)} + \|f\|_{L^p(0,1)}$. Informally $g = f'$. Show that

$$T(t)f(s) = \begin{cases} f(s+t) & s+t < 1 \\ 0 & s+t \geq 1 \end{cases}$$

is a C_0 semi-group on $L^p(0,1)$. Show that its generator is given by $Au = u'$, where $D(A) = \{u \in W_p^1(0,1) : u(1) = 0\}$.