PROJECT IV - MATH 800
DUE APRIL 22, 2021

(1) Problem 6/page 175.
Hint: For $z \in D(0,1)$, take any smooth curve connecting 0 to z, say γ_z. Define
$$g(z) := \int_{\gamma_z} \frac{f'(\xi)}{f(\xi)} d\xi.$$
Show that $g(z)$ is independent on γ_z, just on z. Then, compute $g'(z)$, by computing $\lim_{\delta\to 0} \frac{g(z+\delta)-g(z)}{\delta}$ on appropriate $\gamma_{z+\delta}$.

(2) Problem 8/page 175.

(3) Suppose f is holomorphic in $D(0,1)$ and continuous in $\overline{D(0,1)}$, so that $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions inside $D(0,1)$ of the equation $f(z) = z^n$ for any integer $n \geq 1$.
Hint: Rouche's theorem.

Let $f : D(0,2) \to C$ be holomorphic function. Let $f(0) = 0$ and 0 is a simple zero, i.e. $f'(0) \neq 0$. Also, $f(z) \neq 0$ for any $z \in D(0,1) \setminus \{0\}$. Let $\rho = \min\{|f(z)| : |z| = 1\} > 0$.

(4) Prove that for every $\omega \in D(0,\rho)$, there is an unique $z = z(\omega) \in D(0,1)$, so that $f(z(\omega)) = \omega$.
Hint: Consider
$$k(\omega) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)-\omega} dz, \quad \omega \in D(0,\rho).$$
integrated counterclockwise.

(5) Show that the map $\omega \to z(\omega)$ is holomorphic on $D(0,\rho)$.
Hint: Show that
$$z(\omega) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z f'(z)}{f(z)-\omega} dz, \quad \omega \in D(0,\rho).$$
and prove holomorphicity directly in this formula.

Let f be holomorphic non-constant function in a neighborhood of the unit disc $D(0,1)$. Assume that $|f(z)| = 1$, for $|z| = 1$.

(6) Prove that $f(z) = 0$ has a solution inside $D(0,1)$.
Hint: Use contradiction argument. Apply the maximum modulus principle to both f and $1/f$.

(7) Prove that the image of f contains $D(0,1)$.
Hint: Consider
$$k(\omega) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f'(\xi)}{f(\xi)-\omega} d\xi, \quad \omega \in D(0,1).$$
and use the result from Problem (6).