PROJECT II - MATH 800
DUE MARCH 25TH, 2021

(1) Problem 4/p. 94.
(2) Problem 11 a), d), /p. 95
(3) Problem 17/p. 96
(4) Let f be a holomorphic function on a domain D. Let $z_0 \in D : f(z_0) = 0$. Prove that

$$g(z) := \begin{cases} \frac{f(z)}{(z-z_0)} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

is holomorphic in D.
Note that as a consequence, one can always write $f(z) = (z - z_0)g(z)$ with $g \in H(D)$, whenever $f(z_0) = 0$.
Hint: One way to go is to use Morera's theorem. You can also use a theorem like Theorem 2.3.3. For the Morera'a approach, write

$$\int_\gamma g(z)dz = \int_\gamma g(z)dz + \int_{\Gamma_\epsilon} g(z)dz - \int_{\Gamma_\epsilon} g(z)dz$$

where $\Gamma_\epsilon = \{z : |z - z_0| = \epsilon\}$ traced clockwise. By Cauchy theorem, show that

$$\int_\gamma g(z)dz + \int_{\Gamma_\epsilon} g(z)dz = \int_{\gamma\cup\Gamma_\epsilon} g(z)dz = 0$$

and then show that $\lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} g(z)dz = 0$.
(5) Problem 24/page 97 - Prove that is correct. In fact, show that the radius of convergence is $r' = \min(r, 1)$.
(6) Problem 27/page 97.
Hint: For the case when g has zeros - show first that the function $\frac{f(z)}{g(z)}$ is still well-defined and analytic at the zeros of g. For example, you can argue that it has removable singularities there.
(7) Problem 30/page 98,
Hint: For the polynomial estimate, show that it is impossible by providing a counterexample (say a power series which converges on $\mathbb{C}$, so that $f^{(k)}(0)$ grows faster than any polynomial of k).
(8) 37/page 99
(9) 41/page 100.
(10) 42/page 100.